

# Temperature Equilibration Rate with Fermi-Dirac Statistics

Lowell S. Brown and Robert L. Singleton Jr.

*Los Alamos National Laboratory  
Los Alamos, New Mexico 87545, USA*

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We calculate analytically the electron-ion temperature equilibration rate in a fully ionized, weakly to moderately coupled plasma, using an *exact* treatment of the Fermi-Dirac electrons. The temperature is sufficiently high so that the quantum-mechanical Born approximation to the scattering is valid. It should be emphasized that we do not build a *model* of the energy exchange mechanism, but rather, we perform a systematic first principles *calculation* of the energy exchange. At the heart of this calculation lies the method of dimensional continuation, a technique that we borrow from quantum field theory and use in a novel fashion to regulate the kinetic equations in a *consistent* manner. We can then perform a systematic perturbation expansion and thereby obtain a finite first-principles result to leading and next-to-leading order. Unlike model building, this systematic calculation yields an estimate of its own error and thus prescribes its domain of applicability. The calculational error is small for a weakly to moderately coupled plasma, for which our result is nearly exact. It should also be emphasized that our calculation becomes unreliable for a strongly coupled plasma, where the perturbative expansion that we employ breaks down, and one must then utilize model building and computer simulations. Besides providing new and potentially useful results, we use this calculation as an opportunity to explain the method of dimensional continuation in a pedagogical fashion. Interestingly, in the regime of relevance for many inertial confinement fusion experiments, the degeneracy corrections are comparable in size to the subleading quantum correction below the Born approximation. For consistency, we therefore present this subleading quantum-to-classical transition correction in addition to the degeneracy correction.

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## I. INTRODUCTION AND SUMMARY

We shall calculate the thermal equilibration rate between Fermi-Dirac electrons and Maxwell-Boltzmann ions in a hot, fully ionized plasma. We shall do so *exactly* to leading and next-to-leading order in the plasma number density, and to all orders in the electron fugacity, thereby providing an essentially exact result for weakly to moderately coupled plasmas. We shall work out this problem for two reasons. First, the result is new and is needed in some applications. Second, our previous treatment in Ref. [1] of the plasma stopping power was performed in great generality, and the basic idea behind the dimensional continuation method, a somewhat subtle analytic tool that we employ, may have gotten lost in all the details. We shall use this opportunity of a simpler and specific case to treat the method in a pedagogical fashion and to explain it clearly [2].

Physical systems often contain disparate length or energy scales. For example, plasma physics involves hard collisions at short distances – ultraviolet physics; and soft interactions at large distances entailing collective effects – infrared physics. The resulting interplay of short and long distances produces the familiar Coulomb logarithm. For the electron-ion temperature equilibration rate, and for other such processes involving disparate scales, it is rather easy to calculate the leading contribution, namely the overall factor in front of this logarithm. Although the order of magnitude of this leading order term can usually be obtained from simple dimensional analysis alone, calculating the additional dimensionless factor inside the

logarithm, the sub-leading term, is quite difficult.

A new method [3] employing dimensional continuation has been introduced to deal with such problems, a method that makes the computation of the sub-leading as well as the leading contributions tractable. This method is based on tested principles of quantum field theory constructed over the last fifty years, and it has been used successfully to calculate well measured phenomena such as the Lamb shift [3], often with much more ease than traditional methods. Most recently, the method has been exploited in Ref. [1] by Brown, Preston, and Singleton (BPS) to provide an extensive treatment of the charged particle stopping power in a plasma, the energy loss per unit distance  $dE/dx$  of the charged projectile. One of the topics treated in BPS was the rate at which electrons and ions in a spatially homogeneous plasma, starting with different temperatures, come into thermal equilibrium at a common temperature. Here we shall extend this work to include the case in which the electron fugacity is sufficiently large that the electrons must be treated with a degenerate Fermi-Dirac distribution. This is the case in which Pauli blocking becomes of some importance.

The degeneracy effects that we treat here come into play as the plasma temperature is lowered. We shall calculate the rate for the general case in which the electrons are described by a Fermi-Dirac distribution, with no approximations being made to this distribution. That is, we shall perform the calculation exactly to all orders in the electron fugacity

$$z_e = e^{\beta_e \mu_e}, \quad (1.1)$$

where  $\beta_e = 1/T_e$  is the reciprocal of the electron temperature and  $\mu_e$  is the electron chemical potential. Note that we shall always measure temperature in energy units so that  $\mu_e$  does indeed have the correct units of energy. We shall assume that the plasma is at most moderately coupled, which often implies that the degeneracy corrections are not large. Nonetheless, we shall work out the general case since this is just as easy as treating the case of only mild degeneracy, and the general case may prove to have some application.

A plasma is seldom formed in thermal equilibrium; for example, a laser preferentially heats the light electrons rather than the heavy ions. While a non-equilibrium plasma will of course eventually thermalize, it does so in several stages. First, the electrons rapidly equilibrate among themselves to a common temperature  $T_e$ . Then, somewhat less rapidly, all the various ions equilibrate to a common temperature  $T_i$ . Finally, the electrons and ions begin the process of thermal equilibration, with the electrons delivering their energy density to the ions at a rate

$$\frac{d\mathcal{E}_{eI}}{dt} = -\mathcal{C}_{eI} (T_e - T_i) . \quad (1.2)$$

We shall calculate the rate coefficient  $\mathcal{C}_{eI}$  in the following sections. However, it is useful to examine this result now, the rate coefficient in the form (8.9), which we display below for convenience. This expression for  $\mathcal{C}_{eI}$  is exact to all orders in the electron fugacity, but valid only in a temperature regime in which the short-distance scattering is described by the quantum-mechanical first Born approximation:

$$\mathcal{C}_{eI} = \frac{\omega_I^2}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} \left( \beta_e e^2 \frac{2 e^{\beta_e \mu_e}}{\lambda_e^3} \right) \left\{ \frac{\ln \Lambda}{\exp\{\beta_e \mu_e\} + 1} + \frac{1}{2} \sum_{l=1}^{\infty} (-1)^{l+1} \ln\{l+1\} e^{l\beta_e \mu_e} \right\} , \quad (1.3)$$

where

$$\ln \Lambda = \frac{1}{2} \left[ \ln \left\{ \frac{16\pi}{\kappa_e^2 \lambda_e^2} \right\} - \gamma - 1 \right] . \quad (1.4)$$

Here  $\omega_I^2$  is the sum of the squared ion plasma frequencies  $\omega_i^2$ ,  $m_e$  is the electron mass,  $\lambda_e$  is the electron thermal wave length, and  $\kappa_e^2$  is the electron contribution to the squared Debye wave number, *including* electron degeneracy effects. The precise definition of these quantities is presented in Section II, but we should note here that we employ rationalized Gaussian units, so that the energy of two electrons of charge  $e$  a distance  $r$  apart is given by  $e^2/4\pi r$ . The structure of the first line that appears in our result (1.3) agrees with the previous result of Brysk [4] when his Eq. (35) is re-expressed in terms of our notation. However, Brysk does not obtain the precise result (1.4) for  $\ln \Lambda$ , but rather only an approximate, leading-log evaluation of the usual form  $\ln\{b_{\max}/b_{\min}\}$ .

Moreover, Brysk [4] also does not obtain the second line of our result (1.3). This second line does not contribute in the Maxwell-Boltzmann limit of very small fugacity  $z_e$ , but it does provide a significant first-order correction in  $z_e$ .

The previous work of BPS [1] computed the exact temperature equilibration rate for a weakly to moderately coupled non-degenerate plasma. While this general result was rather complicated, the high-temperature limit, in which the short-distance Coulomb scattering is given by its first Born approximation, has a rather simple form. The non-degenerate rate coefficient of BPS can be obtained from Eq. (1.3) by taking the small-fugacity limit  $z_e \rightarrow 0$ , which gives

$$\mathcal{C}_{eI}^{\text{non-degen}} = \frac{\omega_I^2}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} (\beta_e e^2 n_e) \ln \Lambda_0 , \quad (1.5)$$

where the logarithm  $\ln \Lambda_0$  above is the non-degenerate limit of Eq. (1.4). The term

$$\kappa_{e0}^2 \equiv \beta_e e^2 n_e \quad (1.6)$$

in parentheses follows from the well known relation (2.12) between number density and fugacity, and it is just the square of the non-degenerate form of the electron's Debye wave number. The non-degenerate limit of Eq. (1.4) is accomplished by the substitution  $\kappa_e \rightarrow \kappa_{e0}$ , and expressing this result in terms of the electron plasma frequency provides the from,

$$\ln \Lambda_0 = \frac{1}{2} \left[ \ln \left\{ \frac{8 T_e^2}{\hbar^2 \omega_e^2} \right\} - \gamma - 1 \right] , \quad (1.7)$$

where we have used the relation  $\kappa_{e0}^2 \lambda_e^2 / 2\pi = \hbar^2 \omega_e^2 / T_e^2$ . The rate coefficient of Eq's. (1.5) and (1.7) is just that given previously by Eq. (12.12) of BPS [1], an expression that was also quoted in Eq. (3.61) in the introductory portion of that work. There is, however, one difference between Eq. (1.7) and the result (12.12) of BPS. Namely, the correct term  $-\gamma - 1$  appearing in Eq. (1.7) above was incorrectly written as  $-\gamma - 2$  in Eq's. (12.12) and (3.61) of BPS because of a transcription error in passing between Eq's. (12.43) and (12.44) of BPS. We have taken the opportunity here to correct this mistake.

Equation (1.7) gives the precise definition of the ‘‘Coulomb log’’ (in the non-degenerate limit) for the temperature equilibration process we have been discussing, including the correct additional constant terms, namely the terms  $\ln 8 - \gamma - 1$ . We should, however, emphasize two points. First of all, the ‘‘Coulomb log’’ is by no means a universal quantity, but rather its precise form is process dependent. For example, the ‘‘Coulomb log’’ for electron conductivity differs from the ‘‘Coulomb log’’ for the electron-ion thermal relaxation rate given here [5]. Secondly, we should emphasize that the ‘‘Coulomb log’’ for the relaxation rate does not depend upon the ion temperature  $T_i$  but only upon the electron temperature  $T_e$ . Some authors [5] incorrectly replace the squared electron

Debye wave number  $\kappa_e^2$  by the fully screened Debye wave number  $\kappa_D^2 = \kappa_e^2 + \kappa_I^2$ . This incorrectly includes the ion contribution  $\kappa_I^2$ , and thus introduces a spurious dependence on the ion temperature  $T_I$ .

If the plasma temperature becomes very low, then the population of bound states must be taken into account. In such a regime, our assumption that the plasma is fully ionized and weakly coupled breaks down, and our calculation is no longer reliable. Hence we shall discuss and examine in detail only the limit in which the degeneracy corrections are mild, the regime in which the temperature is not too low and the electron fugacity is not too large. In this limit, only the first-order fugacity contribution from the general result (1.3) need be retained, and therefore the second line in Eq. (1.3), the term omitted by Brysk [4], makes an essential contribution. As the

temperature is lowered, however, the subleading contribution to the Born approximation becomes comparable to the degeneracy correction, and it too must also be accounted for. In Appendix B, we therefore extract this subleading correction in the transition region between quantum and classical scattering from the general result given in BPS. In the following, these two types of corrections will be called the degeneracy correction and the (first) quantum-to-classical transition correction.

Working to leading order in the fugacity, we only need to expand Eq. (1.3) to linear order in  $z_e = e^{\beta_e \mu_e}$ , a result contained in Eq. (8.15). The quantum-to-classical transition correction is contained in Eq. (B33). Expressing the fugacity correction in terms of the density according to Eq. (2.18), the rate coefficient  $\mathcal{C}_{el}$  reads

$$\begin{aligned} \mathcal{C}_{el} \simeq & \frac{\omega_I^2}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} (\beta_e e^2 n_e) \left\{ \frac{1}{2} \left[ \ln \left\{ \frac{8T_e^2}{\hbar^2 \omega_e^2} \right\} - \gamma - 1 \right] \right. \\ & + \frac{n_e \lambda_e^3}{2} \left[ - \left( 1 - \frac{1}{2^{3/2}} \right) \frac{1}{2} \left[ \ln \left\{ \frac{8T_e^2}{\hbar^2 \omega_e^2} \right\} - \gamma - 1 \right] + \left( \frac{1}{2} \ln 2 + \frac{1}{2^{5/2}} \right) \right] \\ & \left. - \frac{\epsilon_H}{T_e} \sum_i \frac{Z_i^2 \omega_i^2}{\omega_I^2} \left[ \zeta(3) \left( \ln \left\{ \frac{T_e}{Z_i^2 \epsilon_H} \right\} - \gamma \right) - 2 \zeta'(3) \right] \right\}, \end{aligned} \quad (1.8)$$

where the numerical values of the zeta-function and its derivative are

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = 1.20205 \dots, \quad (1.9)$$

and

$$\zeta'(3) = - \sum_{k=1}^{\infty} \frac{1}{k^3} \ln k = -0.19812 \dots. \quad (1.10)$$

We also write  $Z_i$  as the ionic charges in units of the electron charge  $e$ . Before examining expression (1.8) in detail, we note that the first line is the leading rate coefficient calculated in BPS [1], Eq. (1.5) above; the second line is the first degeneracy correction following from Eq. (1.3); and the third line is the first quantum-to-classical transition calculated in Appendix B.

In the last line of Eq. (1.8), the ratio  $\epsilon_H/T_e$  describes the relative size of the correction, where

$$\epsilon_H = \left( \frac{e^2}{4\pi} \right)^2 \frac{m_e}{2\hbar^2} \simeq 13.6 \text{ eV} \quad (1.11)$$

is the binding energy of the hydrogen atom. For some temperature and number density regimes of interest, the second and third lines in Eq. (1.8) become comparable

in size. Hence, while our main thrust in this paper is degeneracy corrections, we must also take into account this first quantum-to-classical transition.

It is conventional to write the Coulomb logarithm as  $\ln\{b_{\max}/b_{\min}\}$ , where  $b_{\max}$  is a Debye length long-distance cutoff, while  $b_{\min}$  is a short-distance cutoff that, depending upon the circumstances, is either a classical distance of closest approach  $b_{\text{cl}} \sim e^2/T_e$  or a quantum wave length  $b_{\text{qm}} \sim \lambda_e$ . Often, an interpolation is made [6] between these two limits by writing

$$\begin{aligned} b_{\min}^2 &= b_{\text{cl}}^2 + b_{\text{qm}}^2 = b_{\text{qm}}^2 \left[ 1 + \frac{b_{\text{cl}}^2}{b_{\text{qm}}^2} \right] \\ &\sim b_{\text{qm}}^2 \left[ 1 + \frac{\epsilon_H}{T_e} \right]. \end{aligned} \quad (1.12)$$

Such an interpolation gives a first correction proportional to the proper quantum expansion parameter  $\epsilon_H/T_e$ , but it fails entirely to produce the proper logarithmic behavior  $(\epsilon_H/T_e) \ln\{\epsilon_H/T_e\}$  displayed in the last line of Eq. (1.8).

Figures 1 and 2 illustrate the size of the degeneracy and the first quantum-to-classical transition corrections in an equimolar deuterium-tritium plasma, for electron number densities of  $n_e = 10^{25} \text{ cm}^{-3}$  and  $n_e = 10^{26} \text{ cm}^{-3}$ , respectively. The solid curves denote the size of the degeneracy corrections relative to the leading BPS term, the ra-

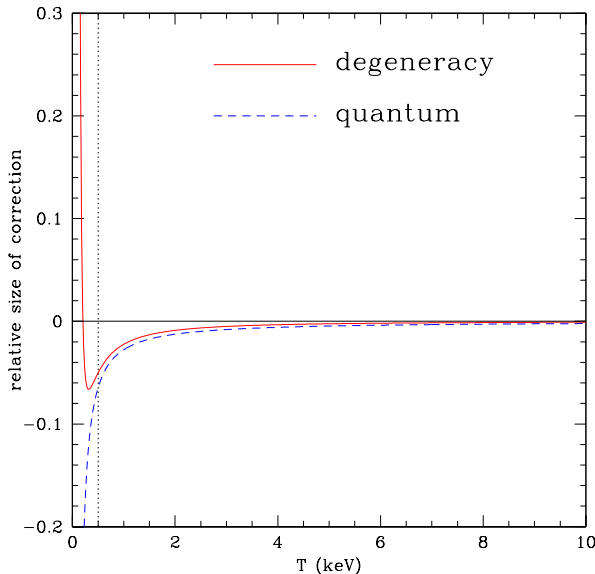


FIG. 1: The relative size of the degeneracy correction and the first classical-to-quantum correction as a function of temperature in keV. The plasma is equimolar deuterium-tritium at an electron number density  $n_e = 10^{25} \text{ cm}^{-3}$ . These corrections are relative to the leading non-degenerate BPS rate: the degeneracy correction (the solid line) is the ratio of the second to the first line in Eq. (1.3), while the classical-to-quantum correction (the dashed line) is the ratio of the third to the first. The electron temperature runs between values 0.1 keV and 10 keV. Our calculation ceases to be valid at low temperatures, and this is indicated by the vertical dotted line.

tio of the second to the first line of Eq. (1.8). The dashed curves show the corresponding size of the first quantum-to-classical transition, the ratio of the third to the first line of Eq. (1.8). Note that both corrections are comparable in magnitude between these two electron number densities. We have plotted the corrections for temperatures between  $T_e = 0.1 \text{ keV}$  and  $T_e = 10 \text{ keV}$ . However, below about 0.5 keV, ionization and other strongly coupled plasma effects become important, and our formalism breaks down. Also, at these lower temperatures, higher order fugacity terms in Eq. (1.4) become important, and this could change the low-temperature behavior of the solid curves; therefore, one should trust Eq. (1.8) only to the right of the vertical dotted line at  $T_e = 0.5 \text{ keV}$ . For densities below  $n_e = 10^{25} \text{ cm}^{-3}$ , the degeneracy correction is much smaller than the quantum-to-classical correction. The situation is reversed for densities greater than  $n_e = 10^{26} \text{ cm}^{-3}$ , where degeneracy effects dominate over the quantum-to-classical corrections. Interestingly, the density and temperature range in which the degeneracy and the quantum-to-classical corrections are comparable lies in the regime relevant for inertial confinement fusion.

We turn now to present the details of our work. After establishing the conventions and notation that we em-

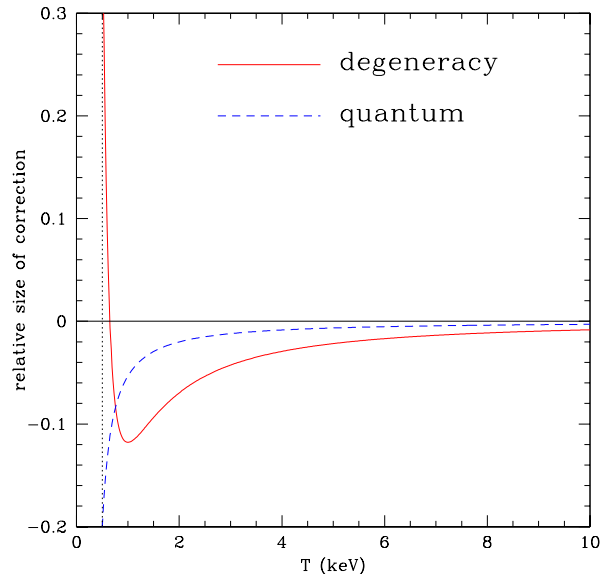


FIG. 2: Same as Fig. 1, except the number density is  $n_e = 10^{26} \text{ cm}^{-3}$ . Again, the solid line is the degeneracy correction and the dashed line is the first quantum-to-classical correction.

ploy, we review and explain in some detail the dimensional continuation method that forms the basis of our calculation, and we relate this method to previous work that utilized convergent kinetic equations. After establishing this foundation, we then perform the calculation of the degeneracy corrections to the rate at which the electrons and ions come into thermal equilibrium. Appendix A contains the details of the plasma dielectric function needed in the text, and Appendix B extracts the first correction in the transition to classical scattering from a general formula given in BPS [1].

But before passing to these details, we would like to make a comment. There are a large number of very able theoretical physicists who work on formal developments that have no contact with experiment. In the chance that one of them may encounter this paper, we would like to encourage them to once in a while apply their skills to topics such as that treated here. The field would profit from their work. In this regard, we quote a penciled poem written in 1957 on note-paper embossed with the header “The White House, Washington”:

*A fact without a theory is like a ship without a sail,  
is like a boat without a rudder,  
is like a kite without a tail.*

*A fact without a theory is like an inconsistent act.  
But if there is one thing worse in this confusing  
universe,  
it is a theory without a fact.*

## II. CONVENTIONS AND NOTATION

We will treat the ions with Maxwell-Boltzmann statistics and the electrons with Fermi-Dirac statistics. The thermal equilibrium form of an ion phase space density  $f_i(\mathbf{p}_i)$  thus reads

$$f_i(\mathbf{p}_i) = e^{-\beta_i (E_i(\mathbf{p}_i) - \mu_i)}, \quad (2.1)$$

while for electrons in thermal equilibrium,

$$f_e(\mathbf{p}_e) = \frac{1}{e^{\beta_e (E_e(\mathbf{p}_e) - \mu_e)} + 1}. \quad (2.2)$$

Letting the index  $b$  refer to either the ions  $i$  or the electrons  $e$ , the kinetic energy is

$$E_b(\mathbf{p}_b) = \frac{p_b^2}{2m_b}, \quad (2.3)$$

and the inverse temperature and chemical potential are

$$\beta_b = 1/T_b \quad \text{and} \quad \mu_b. \quad (2.4)$$

Since we shall work in an arbitrary number of dimensions  $\nu$ , each species number density appears as

$$n_b = \mathbf{g}_b \int \frac{d^\nu p_b}{(2\pi\hbar)^\nu} f_b(\mathbf{p}_b), \quad (2.5)$$

where  $\mathbf{g}_b$  is spin-degeneracy factor. For electrons  $\mathbf{g}_e = 2$ . We are using a notation for the distribution functions  $f_b$  in which the species index  $b$  implicitly includes spin degrees of freedom. It is inconvenient, however, to use this notation for the number density  $n_b$ . Measurements of the species density are usually insensitive to spin degrees of freedom, and we shall therefore denote the number density of the species (including all the spins) by  $n_b$ . This accounts for the factor of  $\mathbf{g}_b$  in Eq. (2.5). For the ions in thermal equilibrium, the integral (2.5) is a product of trivial Gaussian integrals, and so

$$n_i = \mathbf{g}_i \lambda_i^{-\nu} e^{\beta_i \mu_i}, \quad (2.6)$$

where we define the thermal wavelength for species  $b$  as

$$\lambda_b = \hbar \left( \frac{2\pi\beta_b}{m_b} \right)^{1/2}. \quad (2.7)$$

For the electrons in thermal equilibrium, we first pass to hyper-spherical coordinates to write

$$n_e = \mathbf{g}_e \frac{\Omega_{\nu-1}}{(2\pi\hbar)^\nu} \int_0^\infty p^{\nu-1} dp \frac{1}{e^{\beta_e (E_e(\mathbf{p}) - \mu_e)} + 1}. \quad (2.8)$$

Here  $\Omega_{\nu-1}$  is the area of a unit hyper-sphere in a space of  $\nu$  dimensions that is evaluated in the next section with the result

$$\Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)}. \quad (2.9)$$

Changing to dimensionless variables,

$$x = \beta_e E_e(\mathbf{p}) = \beta_e \frac{p^2}{2m_e}, \quad (2.10)$$

allows us to express the electron number density as

$$n_e = \mathbf{g}_e \lambda_e^{-\nu} \frac{1}{\Gamma(\nu/2)} \int_0^\infty \frac{dx}{x} \frac{x^{\nu/2}}{e^{-\beta_e \mu_e} e^x + 1}. \quad (2.11)$$

When the quantity  $-\beta_e \mu_e$  becomes very large (and positive), Fermi-Dirac statistics pass to the Maxwell-Boltzmann limit, with the denominator above becoming a simple exponential. In this limit, the  $x$ -integration becomes the standard representation of  $\Gamma(\nu/2)$ , and we see that the number density in this small fugacity limit is given by the Maxwell-Boltzmann form (2.6), as it must be. Expanding Eq. (2.11) to second order in the fugacity  $z_e = e^{\beta_e \mu_e}$  and using  $\mathbf{g}_e = 2$  gives, for the the physical case of three dimensions,

$$\nu = 3 : \quad n_e \simeq \frac{2}{\lambda_e^3} e^{\beta_e \mu_e} \left[ 1 - \frac{e^{\beta_e \mu_e}}{2^{3/2}} \right]. \quad (2.12)$$

Note that the first correction, which decreases the number density, is simply the fugacity divided by a numerical factor of order unity. By small fugacity, we therefore mean that  $e^{\beta_e \mu_e} \ll 2^{3/2} \simeq 2.8$ .

As we shall see in the following section, in a space of  $\nu$  dimensions, the energy of two charges  $e$  at a distance  $r$  apart is proportional to  $e^2/r^{\nu-2}$ . Since the units of a number density  $n$  are  $(\text{length})^{-\nu}$ , we conclude that  $e^2 n$  has the units of energy over length squared, independently of the spatial dimension  $\nu$ . In particular,

$$\omega_b^2 = \frac{e_b^2 n_b}{m_b} \quad (2.13)$$

is the squared plasma frequency for species  $b$  with the fixed dimension of an inverse-time-squared, regardless of the spatial dimensionality  $\nu$ . The situation for the squared Debye wave number is essentially the same, except that, as noted in Appendix A, in general, this quantity is defined in terms of the fluctuations in the number density, and so

$$\kappa_b^2 = \beta_b e_b^2 \frac{\partial n_b}{\partial(\beta_b \mu_b)}. \quad (2.14)$$

For Maxwell-Boltzmann statistics, the derivative that appears here just reproduces the particle density in accord with the fact that classical particles are described by Poisson statistics. However, for Fermi-Dirac statistics, one must use

$$\frac{\partial n_b}{\partial(\beta_b \mu_b)} = \frac{\mathbf{g}_b \lambda_b^{-\nu}}{\Gamma(\nu/2)} \int_0^\infty \frac{dx}{x} \frac{x^{\nu/2} e^{-\beta_b \mu_b} e^x}{[e^{-\beta_b \mu_b} e^x + 1]^2} \leq n_b. \quad (2.15)$$

The inequality that appears here implies that

$$\kappa_b^2 \leq \beta_b e_b^2 n_b = \kappa_{b0}^2. \quad (2.16)$$

For the dilute case in three dimensions, including the first correction in the fugacity, the electron Debye wave number is given by

$$\begin{aligned} \nu = 3 : \\ \kappa_e^2 &\simeq \beta_e e^2 \frac{2}{\lambda_e^3} e^{\beta_e \mu_e} \left[ 1 - \frac{2}{2^{3/2}} e^{\beta_e \mu_e} \right] \\ &\simeq \beta_e e^2 n_e \left[ 1 - \frac{1}{2^{3/2}} e^{\beta_e \mu_e} \right]. \end{aligned} \quad (2.17)$$

In the first-order fugacity correction that appears here we can use the lowest-order result

$$n_e \simeq \frac{2}{\lambda_e^3} e^{\beta_e \mu_e} \quad (2.18)$$

to compute the fugacity and thus write

$$\nu = 3 : \\ \kappa_e^2 \simeq \beta_e e^2 n_e \left[ 1 - \frac{1}{2^{3/2}} \frac{\lambda_e^3 n_e}{2} \right]. \quad (2.19)$$

### III. METHOD

Since the method of dimensional continuation that we shall use is novel and perhaps subtle, we present here a pedagogical account of its basis.

#### A. Disparate Length Scales; Expansion Parameter

The electron-ion energy exchange brought about by their collisions in a plasma involves a Coulomb interaction that is Debye screened at large distances and, as we shall see in the course of our work, cut off at short distances by quantum effects. As we shall sketch below in Subsection III C for arbitrary spatial dimensions  $\nu$ , the familiar elementary description of this energy transfer for  $\nu = 3$  dimensions involves the impact parameter integral

$$\int_{b_{\min}}^{b_{\max}} \frac{db}{b} = \ln \left\{ \frac{b_{\max}}{b_{\min}} \right\}. \quad (3.1)$$

Here  $b_{\min}$  is the minimum distance of closest approach that, in the quantum limit that is relevant here, is set by the scale of the electron thermal wavelength  $\lambda_e$ . That is,  $b_{\min}$  is some numerical constant times  $\lambda_e$ . The upper limit on the impact parameter integral is set by the electron Debye length  $\kappa_e^{-1}$ , with  $b_{\max}$  some numerical multiple of  $\kappa_e^{-1}$ . Thus

$$\frac{b_{\min}}{b_{\max}} \sim \lambda_e \kappa_e. \quad (3.2)$$

The purpose of the dimensional continuation method is to precisely determine the numerical constants that appear here.

Our method applies when the ratio  $b_{\max}/b_{\min}$  is large: In this case, the dimensionless parameter  $\lambda_e \kappa_e$  is small,

and we shall use it as our expansion parameter. As we noted in the previous section, the Debye wave-number  $\kappa_e$  always has the dimensions of an inverse length, even at arbitrary spatial dimension  $\nu$ . Hence,  $\lambda_e \kappa_e$  is a convenient parameter to employ in our dimensional continuation scheme because it remains dimensionless as the number of spatial dimensions  $\nu$  is varied. Moreover, as we shall see, it is the combination  $\lambda_e \kappa_e$  that directly arises as our computations progress.

The electron plasma coupling strength is characterized by the ratio of the Coulomb electrostatic energy of two electrons a Debye distance apart divided by the temperature. In the physical space of three dimensions,  $\nu = 3$ , this is the dimensionless parameter

$$\nu = 3 : \quad g_e = \beta_e \frac{e^2 \kappa_e}{4\pi}. \quad (3.3)$$

The perturbative expansion of plasma thermodynamic parameters involve a series of ascending *integer* powers (up to additional logarithmic corrections) of the coupling constant  $g_e$ . Except for different conventions that can alter a trivial overall factor, the electron quantum Coulomb parameter is defined as the Coulomb energy for two electrons a thermal wave length apart divided by the temperature. For three dimensions, this reads

$$\nu = 3 : \quad \eta_e = \beta_e \frac{e^2}{4\pi \lambda_e}. \quad (3.4)$$

Hence, in three dimensions,

$$\nu = 3 : \quad \lambda_e \kappa_e = \frac{g_e}{\eta_e}. \quad (3.5)$$

Thus our expansion parameter  $\lambda_e \kappa_e$  is essentially the plasma coupling parameter  $g_e$ , albeit divided by the quantum parameter  $\eta_e$ . Accordingly, one could equivalently work in terms of the coupling  $g_e$  as we have done in the past [1], but here it is more convenient to use  $\lambda_e \kappa_e$ , and so this we shall do.

Our work applies to fully ionized plasmas where the temperature is large and thus the parameter  $\eta_e$  is small. The condition that  $\lambda_e \kappa_e$  be small requires that the plasma coupling  $g_e$  be even smaller than  $\eta_e$ . To put this in perspective, we recall that even if Fermi-Dirac statistics are required, the Debye wave-number is smaller than that given by the Maxwell-Boltzmann form with the same temperature and density. Hence we have

$$\lambda_e^2 \kappa_e^2 \leq \lambda_e^2 \beta_e e^2 n_e, \quad (3.6)$$

where the electron number density on the right-hand side of this equation is given by Maxwell-Boltzmann statistics. Using now the number density (2.6) in the Maxwell-Boltzmann limit and the definition (2.7) of the thermal wavelength, we find that

$$\begin{aligned} \lambda_e^2 \beta_e e^2 n_e &= 8 \pi^{1/2} e^{\beta_e \mu_e} \sqrt{\frac{\epsilon_H}{T_e}} \\ &= 8 \pi^{1/2} e^{\beta_e \mu_e} \sqrt{\frac{13.6 \text{ eV}}{T_e}}. \end{aligned} \quad (3.7)$$

Thus, even for somewhat large electron fugacities  $e^{\beta_e \mu_e}$ , the expansion parameter  $\lambda_e \kappa_e$  will be small provided the temperature is reasonably large.

To explain further the utility of  $\lambda_e \kappa_e$  as the appropriate expansion parameter, we examine the situation when the spatial dimension  $\nu$  departs from its physical value  $\nu = 3$ . In this case, as we shall soon see in the next subsection, the Coulomb potential a distance  $r$  away from a point charge has the dependence  $r^{-(\nu-2)}$ . Thus the plasma coupling and quantum Coulomb parameters have the form

$$g_e \sim \beta_e e^2 \kappa_e^{\nu-2}, \quad \eta_e \sim \beta_e \frac{e^2}{\lambda_e^{\nu-2}}, \quad (3.8)$$

and so  $g_e/\eta_e \sim (\lambda_e \kappa_e)^{\nu-2}$  or

$$\lambda_e \kappa_e \sim \left( \frac{g_e}{\eta_e} \right)^{1/(\nu-2)}. \quad (3.9)$$

This emphasizes that although the form of the coupling  $\lambda_e \kappa_e$  that we employ here does not change as the spatial dimension is altered, its form in terms of  $g_e/\eta_e$  does depend upon this dimensionality.

## B. Idea of Dimensional Continuation

We have already seen explicitly how a geometrical quantity, namely the number density, can be computed in a space of arbitrary dimensionality  $\nu$ . In fact, all fundamental theories can be formulated in a world that has space of arbitrary dimensionality. Modern quantum field theory, the mother of all physical theory, is generally formulated for spaces of arbitrary dimensionality in order to regulate it. [See, for example, Ref. [8].] The well known BBGKY hierarchy of coupled equations that depicts general kinetics can obviously be written in a space of arbitrary dimensionality  $\nu$ . For  $\nu > 3$ , the Coulomb force acts as a short-range force; for  $\nu < 3$ , it acts a long-range force. Although the complete BBGKY set of coupled equations for Coulomb forces must remain valid for arbitrary spatial dimensionality  $\nu$ , it cannot be approximated by the Boltzmann or Lenard-Balescu equations for general  $\nu$  values. To leading order in the density, the Boltzmann equation describes the short-distance, hard scattering correctly while the Lenard-Balescu equation correctly describes the long-distance, collective interactions. Hence, to this leading order, the BBGKY hierarchy of equations reduces to the Boltzmann equation for  $\nu > 3$ , but for  $\nu < 3$ , the BBGKY hierarchy reduces to the Lenard-Balescu equation. We shall see how this works out in detail as our work progresses.

Here we introduce the idea of dimensional continuation by examining the case of electrostatics. The Poisson equation for a point charge  $Ze$  in  $\nu$  dimensions reads

$$-\nabla^2 \phi^{(\nu)}(\mathbf{r}) = Ze \delta^{(\nu)}(\mathbf{r}). \quad (3.10)$$

Its solution may be expressed as a Fourier integral:

$$\phi^{(\nu)}(\mathbf{r}) = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{Ze}{\mathbf{k}^2} e^{i\mathbf{k} \cdot \mathbf{r}}. \quad (3.11)$$

As it stands, this integral is defined for all positive integer dimensions  $\nu$ . As is well known from the theory of complex functions, an analytic function is defined from its values on the positive real integers, provided that the function does not diverge rapidly at infinity in the complex plane [7]. But for our equations, we can obtain this extension by explicit calculations. For the case at hand, we first write

$$\frac{1}{\mathbf{k}^2} \equiv \frac{1}{k^2} = \int_0^\infty ds e^{-k^2 s}, \quad (3.12)$$

and interchange integrals to encounter

$$\begin{aligned} & \int \frac{d^\nu k}{(2\pi)^\nu} e^{-k^2 s} e^{i\mathbf{k} \cdot \mathbf{r}} \\ &= \int \frac{d^\nu k}{(2\pi)^\nu} \exp \left\{ - \left[ \mathbf{k} - i \frac{\mathbf{r}}{2s} \right]^2 s \right\} \exp \left\{ - \frac{r^2}{4s} \right\} \\ &= \left( \frac{1}{4\pi s} \right)^{\nu/2} \exp \left\{ - \frac{r^2}{4s} \right\}. \end{aligned} \quad (3.13)$$

Here we have completed the square and used the variable in the square brackets as the new integration variable to obtain a product of  $\nu$  ordinary Gaussian integrals whose evaluation produces the final result. The change of variables from  $s$  to  $t = r^2/4s$  now gives

$$\begin{aligned} \phi^{(\nu)}(\mathbf{r}) &= \frac{Ze}{r^{\nu-2}} \left( \frac{1}{\pi} \right)^{\nu/2} \frac{1}{4} \int_0^\infty \frac{dt}{t} t^{(\nu-2)/2} e^{-t} \\ &= \frac{Ze}{r^{\nu-2}} \left( \frac{1}{\pi} \right)^{\nu/2} \frac{1}{4} \Gamma \left( \frac{\nu-2}{2} \right), \end{aligned} \quad (3.14)$$

since the  $t$  integral is a standard representation of the gamma function. This result now defines an electrostatic potential for any value of  $\nu$  in the entire complex plane.

As a mathematical application of this result, we note that it gives the electric field

$$\mathbf{E}(\mathbf{r}) = \frac{Ze \hat{\mathbf{r}}}{r^{\nu-1}} \left( \frac{1}{\pi} \right)^{\nu/2} \frac{1}{2} \Gamma \left( \frac{\nu}{2} \right). \quad (3.15)$$

Hence Gauss' law applied to a sphere of radius  $r$ ,

$$S(r) \hat{\mathbf{r}} \cdot \mathbf{E}(\mathbf{r}) = Ze, \quad (3.16)$$

informs us that this sphere has an area  $S = \Omega_{\nu-1} r^{\nu-1}$  where

$$\Omega_{\nu-1} = \frac{2 \pi^{\nu/2}}{\Gamma(\nu/2)} \quad (3.17)$$

is the area of a  $(\nu-1)$ -dimensional unit hypersphere embedded in the  $\nu$ -dimensional space.

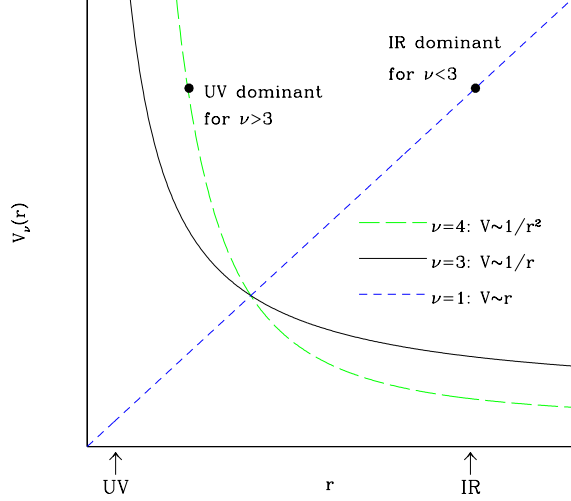


FIG. 3: Short-distance or ultraviolet (UV) physics dominates in dimensions  $\nu > 3$ . Long-distance or infrared (IR) physics dominates when  $\nu < 3$ . UV and IR physics are equally important in  $\nu = 3$ .

There are physical implications that follow from (3.14) of the electrostatic potential of a point charge in  $\nu$  dimensions. These are brought out in Fig. 3. As the figure shows, the Coulomb potential of a point charge becomes more singular at the origin as the spatial dimension  $\nu$  increases: the physics at short distances is increasingly emphasized as the spatial dimension  $\nu$  increases. Since short distances correspond to high wave numbers, this is equivalent to stating that large  $\nu$  emphasizes ultraviolet physical processes. Conversely, as the spatial dimension  $\nu$  decreases, the potential falls off less rapidly at large distances: the physics at large distances becomes ever more important as the spatial dimension  $\nu$  decreases. Since long distances correspond to low wave numbers, this is equivalent to stating that small  $\nu$  emphasizes infrared physical processes. As we shall see, the electron-ion energy exchange can be computed with the Boltzmann equation for  $\nu > 3$  since it correctly accounts for hard scattering. The result, however, has a simple pole that diverges as  $\nu \rightarrow 3$  from above. Conversely, the electron-ion energy exchange can be computed from the Lenard-Balescu equation for  $\nu < 3$  since it correctly accounts for the long-range screening brought about by the collective, dielectric effects in the plasma. The result, however, has a simple pole and diverges as  $\nu \rightarrow 3$  from below. These general features are brought out in the simple computation of the next subsection.

### C. Energy Loss Structure in $\nu$ Spatial Dimensions

To illustrate the remarks that we have been making, we consider the lowest-order energy loss of an electron

passing by a fixed point of charge  $Ze$ . In zeroth order, an electron with impact parameter  $\mathbf{b}$  and perpendicular velocity  $\mathbf{v}$ , so that  $\mathbf{b} \cdot \mathbf{v} = 0$ , simply follows the straight line  $\mathbf{b} + \mathbf{v}t$  as a function of the time  $t$ . In first approximation, with the electric field  $\mathbf{E}$  given by Eq. (3.15), the electron acquires a momentum transfer

$$\Delta \mathbf{p} = -e \int_{-\infty}^{+\infty} dt \mathbf{E}(\mathbf{b} + \mathbf{v}t) \quad (3.18)$$

in passing by the fixed point charge  $Ze$ . This gives an energy change

$$\Delta E = \frac{\Delta \mathbf{p}^2}{2m_e}. \quad (3.19)$$

It is a straightforward matter to check that this gives, up to numerical factors of no importance,

$$\Delta E \sim \frac{1}{m_e} \left( \frac{Ze^2}{v} \right)^2 \left( \frac{1}{b^{\nu-2}} \right)^2. \quad (3.20)$$

In  $\nu$  spatial dimensions, an element of cross section is given by  $d\sigma = \Omega_{\nu-2} b^{\nu-2} db$ . Hence, the weighted energy loss has the form

$$\int d\sigma \Delta E \sim \int_{b_{\min}}^{b_{\max}} \frac{db}{b^{\nu-2}}. \quad (3.21)$$

This example explicitly demonstrates that large  $\nu$  is dominated by short-distance physics and small  $\nu$  is dominated by long-distance physics. Moreover, it shows explicitly that  $\nu = 3$  is the dividing line between these two regions. To bring this out, all we need do is to note that for  $\nu > 3$  the impact parameter integral is not sensitive to the large distance cut off, and we may take the limit  $b_{\max} \rightarrow \infty$  to obtain

$$\nu > 3 : \quad I^>(\nu) = \int_{b_{\min}}^{\infty} \frac{db}{b^{\nu-2}} = \frac{b_{\min}^{3-\nu}}{\nu-3}. \quad (3.22)$$

Conversely, for  $\nu < 3$ , we may set  $b_{\min} = 0$ , with

$$\nu < 3 : \quad I^<(\nu) = \int_0^{b_{\max}} \frac{db}{b^{\nu-2}} = \frac{b_{\max}^{3-\nu}}{3-\nu}. \quad (3.23)$$

The results displayed are the dominant forms in the two different regions of spatial dimensionality  $\nu$ .

### D. Implementation of Dimensional Continuation

The situation that we have just described leads to a well defined result because it is akin to the following example. Suppose that we have a theory that is well defined in the neighborhood of the physical dimension  $\nu = 3$ , and that the theory contains a small parameter  $\epsilon$ . Moreover, suppose that we need to evaluate a function  $F$  that depends upon this small parameter  $\epsilon$  in the following fashion. For  $\nu > 3$  the leading behavior of  $F$  goes like  $\epsilon^{3-\nu}$ ,



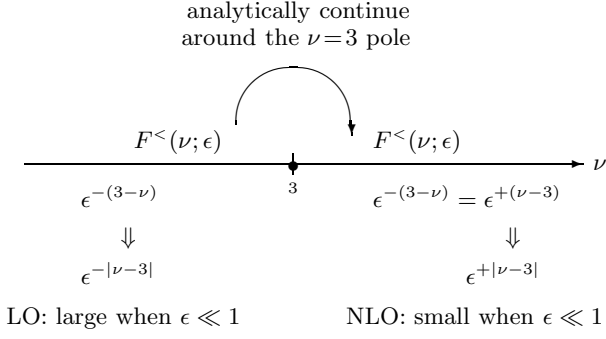


FIG. 4: The analytic continuation of  $F^<(\nu; \epsilon)$  from  $\nu < 3$  to the region  $\nu > 3$ : the same expression can be used for  $F^<(\nu; \epsilon)$  throughout the complex plane since the pole at  $\nu = 3$  can easily be avoided. Note that the quantity  $F^<(\nu; \epsilon) \sim \epsilon^{(\nu-3)}$  is leading order in  $\epsilon$  for  $\nu < 3$ . However, upon analytically continuing to  $\nu > 3$  we find that  $F^<(\nu; \epsilon) \sim \epsilon^{|\nu-3|}$  which is next-to-leading order in  $\epsilon$  relative to  $F^>(\nu; \epsilon) \sim \epsilon^{-|\nu-3|}$ .

and the function  $F$  has a simple pole in  $\nu$  as  $\nu \rightarrow 3$  from above. Conversely, for  $\nu < 3$ , the leading behavior of the function  $F$  goes like  $\epsilon^{\nu-3}$  and the function  $F$  has a simple pole in  $\nu$  as  $\nu \rightarrow 3$  from below. That is, we have the leading terms

$$\nu > 3 : \quad F^>(\nu; \epsilon) = A^>(\nu) \epsilon^{3-\nu}, \quad (3.24)$$

and

$$\nu < 3 : \quad F^<(\nu; \epsilon) = A^<(\nu) \epsilon^{\nu-3}. \quad (3.25)$$

Since the two contributions each have poles in  $\nu$ ,

$$A^>(\nu) = \frac{R^>}{\nu-3} + r^> + \mathcal{O}(\nu-3), \quad (3.26)$$

and

$$A^<(\nu) = \frac{R^<}{3-\nu} + r^< + \mathcal{O}(3-\nu). \quad (3.27)$$

The function  $F^<(\nu; \epsilon)$  is of leading order in the expansion parameter  $\epsilon$  for  $\nu < 3$ . Since it is an analytic function of  $\nu$ , it may be continued throughout the complex  $\nu$ -plane. When it is analytically continued to  $\nu > 3$  it becomes of subleading order. This behavior is depicted in Fig. 4. Exactly the converse situation applies to the function  $F^>(\nu; \epsilon)$ .

Therefore, in the neighborhood of  $\nu = 3$ ,

$$\begin{aligned} F(\nu; \epsilon) &= F^>(\nu; \epsilon) + F^<(\nu; \epsilon) \\ &= A^>(\nu) \epsilon^{3-\nu} + A^<(\nu) \epsilon^{\nu-3} \end{aligned} \quad (3.28)$$

is accurate to leading and sub-leading order in  $\epsilon$ . For  $\nu > 3$ , the term with the coefficient  $A^>(\nu)$  is dominant while that with the coefficient  $A^<(\nu)$  is sub-dominant. For  $\nu < 3$  the roles of the dominant and sub-dominant terms are interchanged. It should be emphasized that the addition of the two terms contains no double counting

since in each region one term, and one term only, dominates. Since the theory is well defined at the physical dimension  $\nu = 3$ , the poles must cancel, which requires that

$$R^> = R^<. \quad (3.29)$$

Using

$$\epsilon^{\pm(\nu-3)} = e^{\pm(\nu-3) \ln \epsilon}, \quad (3.30)$$

we now have, in the neighborhood of  $\nu = 3$ ,

$$\begin{aligned} F(\nu; \epsilon) &= \frac{R^>}{\nu-3} \left[ e^{+(\nu-3) \ln \epsilon} - e^{-(\nu-3) \ln \epsilon} \right] + r^> + r^< \\ &= 2R^> \ln \epsilon + (r^> + r^<). \end{aligned} \quad (3.31)$$

We must emphasize that this method of dimensional continuation provides not only the coefficient  $2R$  out in front of  $\ln \epsilon$  (which is often not too difficult to compute), but the constant  $r^> + r^<$  in addition to this logarithm (which is often difficult to compute).

A relevant example is provided by the simple model of the energy loss presented in the previous subsection. According to our general method, in the neighborhood of  $\nu = 3$  we must have

$$\begin{aligned} I(\nu) &= I^>(\nu) + I^<(\nu) \\ &= \frac{b_{\min}^{3-\nu}}{\nu-3} + \frac{b_{\max}^{3-\nu}}{3-\nu}. \end{aligned} \quad (3.32)$$

The  $\nu \rightarrow 3$  limit produces

$$\begin{aligned} I(\nu) &= \frac{b_{\max}^{3-\nu}}{\nu-3} \left[ \left( \frac{b_{\min}}{b_{\max}} \right)^{3-\nu} - 1 \right] \\ &\rightarrow -\ln \left\{ \frac{b_{\min}}{b_{\max}} \right\}. \end{aligned} \quad (3.33)$$

This is precisely the value (3.1) of the familiar impact integral evaluated directly in three dimensions

Another instructive example of this method is provided by an examination of the modified Bessel function  $K_\nu(z)$  for small  $\nu$  and small  $z$ . This is discussed in Ref's. [3] and [1], and in more detail in Ref. [2]. These works should be consulted if the explanation already given is not convincing.

Although we have sketched the basic idea of our dimensional continuation scheme leading to the result (3.31), in this paper we shall apply it in a slightly different form, a form similar to the more physical example that led to the result (3.33). As we have explained, the electrostatic potential in  $\nu$  spatial dimensions has the functional form  $e/r^{\nu-2}$  so that the energy between two point charges a distance  $r$  apart is proportional to  $e^2/r^{\nu-2}$ . As we shall see explicitly in our work below, the electron-ion energy exchange rate contains an over all dimension-bearing factor  $\beta_e e^2$ . This factor has the dimensions of length to the

power  $\nu - 2$ . To define a quantity whose physical dimension does not vary as the spatial dimension varies, the factor  $\beta_e e^2$  must be accompanied by a factor of length raised to the power  $3 - \nu$ , which gives a result that has a constant factor of 1/length in all spatial dimensions  $\nu$ . For the  $\nu > 3$  contribution, a scattering term with a length cutoff given by the electron thermal wavelength  $\lambda_e$ , the needed dimensional factor is given by  $\lambda_e^{3-\nu}$  as we shall explicitly find below. For the  $\nu < 3$  contribution, a long-distance Debye screened interaction term, the needed dimensional factor is given by  $(1/\kappa_e)^{3-\nu}$  as we shall also explicitly see below. Thus, in all spatial dimensions near  $\nu = 3$ , the rate has the structure

$$G(\nu) = \beta_e e^2 \left[ \lambda_e^{3-\nu} B^>(\nu) + \left( \frac{1}{\kappa_e} \right)^{3-\nu} B^<(\nu) \right], \quad (3.34)$$

and for  $\nu$  near  $\nu = 3$ ,

$$B^>(\nu) = \frac{R}{\nu - 3} + b_> \quad (3.35)$$

and

$$B^<(\nu) = \frac{R}{3 - \nu} + b_<. \quad (3.36)$$

Writing  $B = b_> + b_<$ , we find that for  $\nu$  near  $\nu = 3$ ,

$$\begin{aligned} G(\nu) &= \beta_e e^2 \lambda_e^{3-\nu} \left[ \frac{R}{\nu - 3} \left\{ 1 - (\lambda_e \kappa_e)^{\nu-3} \right\} + B \right] \\ &\rightarrow \beta_e e^2 \left[ -R \ln\{\lambda_e \kappa_e\} + B \right], \end{aligned} \quad (3.37)$$

in which the final line gives the  $\nu = 3$  limit.

An objection could be raised that we have not shown explicitly that larger sub-leading terms are not present. We have extracted terms that have the generic behavior  $\epsilon^{3-\nu}$  for  $\nu > 3$  and  $\epsilon^{\nu-3}$  for  $\nu < 3$ . One might ask if there are additional terms with a power law dependence between  $\epsilon^{3-\nu}$  and  $\epsilon^{\nu-3}$ . However, simple dimensional analysis shows that such terms of intermediate order cannot appear. The point is that the physics involves *only two* different mechanisms that dominate at large and small scales. These two different mechanisms involve different combinations of basic physical parameters and hence give quite different dependencies on the small parameter when the spatial dimension  $\nu$  departs from  $\nu = 3$ .

#### IV. CONVERGENT KINETIC EQUATIONS

A number of authors [9, 10, 11, 12] have proposed various versions of plasma kinetic equations that have neither short nor long range divergences. This work is summarized in the book of Liboff [13], which we shall outline here and then relate to our method of dimensional regularization.

Liboff, in his Eq. (2.75), writes the transport equation for a homogeneous system such as we consider as

$$\frac{\partial f}{\partial t} = B_0 + L_0 - \bar{R}, \quad (4.1)$$

where  $B_0$  is the Boltzmann collision integral,  $L_0$  the Lenard-Balescu integral, and  $\bar{R}$  is a renormalization term that cancels the singularities in  $B_0$  and  $L_0$  [14]. We should note that starting off with admittedly infinite, and therefore undefined quantities, as in Ref. [13], is at best a heuristic procedure. This is to be contrasted with the renormalization procedure performed in modern quantum field theories where the starting point is a rigorously defined, finite theory because the starting point is a *regularized* theory. At any rate, the infinite renormalization term  $\bar{R}$  is expressed formally as a double integral over both impact parameters and Fourier wave numbers. The integral over impact parameters  $b$  is broken up into a large impact parameter part  $b > b_0$  and a small impact parameter part  $b < b_0$ ,  $\bar{R} = \bar{R}( > b_0 ) + \bar{R}( < b_0 )$ . It is then shown that  $\bar{R}$  has a formal construction such that both  $B_0 - \bar{R}( > b_0 )$  and  $L_0 - \bar{R}( < b_0 )$  are finite.

The transport equation is thus rendered finite. Liboff concludes, “So we find that the combination of collision integrals gives a reasonable model for a convergent plasma kinetic equation.” Our goal, however, is not just to find a “reasonable model,” but to *calculate* the Coulomb logarithm in a precise and rigorous fashion. To leading order in the plasma density, we shall not only compute the coefficient out in front of the logarithm, but also the constants that appear in addition to the logarithm. See Ref. [2] for more details.

Although Gould and DeWitt [11] also separate the right-hand side of the transport equation into three terms, they do so in such a fashion that each of the terms is finite and well defined. As shown in Appendix B of BPS [1], the formulation of Gould and DeWitt correctly gives the constant term as well as the leading Coulomb logarithm, and as far as these terms are concerned, their work is mathematically equivalent to our method of dimensional continuation. Both are accurate to  $\mathcal{O}(g^2)$  in the plasma coupling, and no better. The trouble with their formulation is that it also produces a subset of higher order terms, and there is no reason that these provide a more accurate evaluation. As is well known, the inclusion of partial subsets of higher-order terms can sometimes give less rather than more accurate results.

#### V. ENERGY AND TEMPERATURE RATES

The rate of change in the electron energy density transported to all the ions species vanishes when the two subsystems have the same temperature. Hence we may write

$$\frac{d\mathcal{E}_{eI}}{dt} = -\mathcal{C}_{eI} (T_e - T_i). \quad (5.1)$$

Since energy flows from the electrons to the ions when the electrons are hotter than the ions,  $\mathcal{C}_{eI}$  is positive. Since

the total energy is conserved, the rate at which energy is transferred from the ions to the electrons,  $d\mathcal{E}_{Ie}/dt$ , has the same coefficient  $\mathcal{C}_{eI}$  but an overall sign change or, equivalently,

$$\frac{d\mathcal{E}_{Ie}}{dt} = -\mathcal{C}_{eI} (T_I - T_e). \quad (5.2)$$

A change in the energy of a subsystem in the plasma produces a corresponding change in the temperature of that subsystem. Thus, for the electrons,

$$\Delta\mathcal{E}_{eI} = c_e \Delta T_e, \quad (5.3)$$

while for the ions

$$\Delta\mathcal{E}_{Ie} = \sum_i \Delta\mathcal{E}_i = c_I \Delta T_I. \quad (5.4)$$

Here, since the plasma interactions do not change particle number densities, the specific heats  $c_e$  and  $c_I$  are those at constant volume. Since  $\Delta\mathcal{E}_{eI}$  is an energy density, these are the specific heats per unit volume. For a hot plasma that is not strongly coupled, the case treated in this paper, these specific heats are given by the familiar ideal gas results:

$$c_e = 3n_e/2 \quad \text{and} \quad c_I = 3n_I/2, \quad (5.5)$$

where  $n_I$  is the total ionic density, the number of all the ions per unit volume. Thus Eq's. (5.1) and (5.2) are equivalent to

$$\frac{dT_e}{dt} = -\gamma_{eI} (T_e - T_I), \quad (5.6)$$

with  $\gamma_{eI} = \mathcal{C}_{eI}/c_e$ ; and

$$\frac{dT_I}{dt} = -\gamma_{Ie} (T_I - T_e), \quad (5.7)$$

with  $\gamma_{Ie} = \mathcal{C}_{eI}/c_I$ . Moreover, the rate at which the separate temperatures approach one another is given by

$$\frac{d(T_e - T_I)}{dt} = -\Gamma (T_e - T_I), \quad (5.8)$$

in which

$$\Gamma = \mathcal{C}_{eI} \left( \frac{1}{c_e} + \frac{1}{c_I} \right). \quad (5.9)$$

We turn now to compute the rate coefficient  $\mathcal{C}_{eI}$ .

## VI. BOLTZMANN EQUATION: SHORT-DISTANCE PHYSICS

We first work in  $\nu > 3$  dimensions where the short-distance physics dominates. Thus the rate of change of the electron distribution is described by the Boltzmann equation with Fermi-Dirac statistics for the electrons and Maxwell-Boltzmann statistics for the heavy ions. The Boltzmann equation for the electron distribution, including the Pauli blocking of the scattered electrons and two-body quantum scattering effects, reads

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$$\begin{aligned} \frac{\partial f_e(\mathbf{p}_e)}{\partial t} = & \sum_i \int \frac{d^\nu p'_e}{(2\pi\hbar)^\nu} \frac{d^\nu p'_i}{(2\pi\hbar)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu} |T|^2 \\ & (2\pi\hbar)^\nu \delta^\nu(\mathbf{p}'_e + \mathbf{p}'_i - \mathbf{p}_e - \mathbf{p}_i) (2\pi\hbar) \delta\left(\frac{\mathbf{p}'_e{}^2}{2m_e} + \frac{\mathbf{p}'_i{}^2}{2m_i} - \frac{\mathbf{p}_e^2}{2m_e} - \frac{\mathbf{p}_i^2}{2m_i}\right) \\ & \left\{ f_e(\mathbf{p}'_e) f_i(\mathbf{p}'_i) [1 - f_e(\mathbf{p}_e)] - f_e(\mathbf{p}_e) f_i(\mathbf{p}_i) [1 - f_e(\mathbf{p}'_e)] \right\}, \end{aligned} \quad (6.1)$$

where  $T$  is the amplitude for the two-body scattering collision  $e i \rightarrow e' i'$ , and we have omitted the spatial convection term on the left-hand side because we are concerned with spatially uniform plasmas. Since the electrons are in thermal equilibrium with each other, there is no electron-electron interaction contribution to this time derivative. The electron kinetic energy density — the electron energy per unit volume — has the same form as the number density (2.5) save that an additional factor of  $E_e(\mathbf{p}) = p^2/2m_e$  appears in the integrand. Hence the rate at which this energy density changes because of the electron ion interactions is given by

$$\frac{\partial \mathcal{E}_{eI}}{\partial t} = 2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{p_e^2}{2m_e} \frac{\partial f_e(\mathbf{p}_e)}{\partial t}, \quad (6.2)$$

where the factor of 2 multiplying the integral accounts for the electron spin degeneracy. Using the crossing symmetry  $\mathbf{p}_e \leftrightarrow \mathbf{p}'_e$  and  $\mathbf{p}_i \leftrightarrow \mathbf{p}'_i$  of the scattering amplitude  $T$  in (6.1), the rate of energy exchange from the electrons to the

ions (6.2) can be written as

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^>}{\partial t} = & 2 \sum_i \int \frac{d^\nu p'_e}{(2\pi\hbar)^\nu} \frac{d^\nu p'_i}{(2\pi\hbar)^\nu} \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu} |T|^2 \\ & (2\pi\hbar)^\nu \delta^\nu(\mathbf{p}'_e + \mathbf{p}'_i - \mathbf{p}_e - \mathbf{p}_i) (2\pi\hbar) \delta\left(\frac{\mathbf{p}'_e{}^2 - \mathbf{p}_e^2}{2m_e} + \frac{\mathbf{p}'_i{}^2 - \mathbf{p}_i^2}{2m_i}\right) \\ & \frac{\mathbf{p}'_e{}^2 - \mathbf{p}_e^2}{2m_e} f_e(\mathbf{p}_e) f_i(\mathbf{p}_i) [1 - f_e(\mathbf{p}'_e)], \end{aligned} \quad (6.3)$$

where the factor of two in front of the sum is the spin-degeneracy  $g_e = 2$  for electrons. We have placed a “greater-than” superscript on the left-hand side of the equation since we are now computing the  $\nu > 3$  contribution. We start by performing the  $\mathbf{p}'_i$ -integration in Eq. (6.3), using the momentum conserving delta-function to set

$$\mathbf{p}'_i = \mathbf{p}_e + \mathbf{p}_i - \mathbf{p}'_e. \quad (6.4)$$

Defining the momentum transfer by

$$\mathbf{q} \equiv \mathbf{p}'_e - \mathbf{p}_e = \mathbf{p}_i - \mathbf{p}'_i, \quad (6.5)$$

and the average of the initial and final electron momenta by

$$\bar{\mathbf{p}} \equiv \frac{1}{2} [\mathbf{p}'_e + \mathbf{p}_e], \quad (6.6)$$

we can simplify Eq. (6.3) to read

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^>}{\partial t} = & 2 \sum_i \int \frac{d^\nu p'_e}{(2\pi\hbar)^\nu} \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu} |T|^2 (2\pi\hbar) \delta\left(\frac{1}{m_i} \mathbf{p}_i \cdot \mathbf{q} - \frac{1}{m_e} \bar{\mathbf{p}} \cdot \mathbf{q} - \frac{1}{2m_i} \mathbf{q}^2\right) \\ & \frac{1}{m_e} \bar{\mathbf{p}} \cdot \mathbf{q} f_e(\mathbf{p}_e) f_i(\mathbf{p}_i) [1 - f_e(\mathbf{p}'_e)]. \end{aligned} \quad (6.7)$$

Since  $T$  is a two-body scattering amplitude, its general form can depend upon both the square of the momentum transfer  $q^2 = \mathbf{q} \cdot \mathbf{q}$  and the total center-of-mass energy  $W = p^2/2m_{ei}$ , where the relative momentum is given by  $\mathbf{p} = m_{ei}(\mathbf{v}_e - \mathbf{v}_i)$ , with  $m_{ei}$  being the reduced electron-ion mass. It is the  $W$ -dependence in  $T = T(W, q^2)$  that renders the integrals in Eq. (6.7) difficult to calculate because  $W$  depends explicitly on  $\mathbf{p}_i$ . In Section 12 of Ref. [1], this calculation is performed to all orders in the Coulomb scattering. For the work here, we shall be less general and exploit the fact that the electron-ion mass ratio  $m_e/m_i$  is very small (so the reduced mass  $m_{ei}$  is almost equal to the electron mass  $m_e$ ). We shall assume that the electron and ion temperatures are not orders of magnitude apart, a mild restriction in all practical applications, so that

$$\beta_e m_e \ll \beta_i m_i. \quad (6.8)$$

Under these circumstances, the thermal average electron velocity is much larger than the ion velocity, and to a very good approximation  $|\mathbf{v}_e - \mathbf{v}_i| = |\mathbf{v}_e|$ . Thus the quantum Coulomb parameter  $\eta_{ei} = e e_i / 4\pi\hbar |\mathbf{v}_e - \mathbf{v}_i|$  that appears in the Boltzmann equation can be replaced by a Coulomb parameter that contains only the electron velocity,  $\eta_{ei} \rightarrow Z_i e^2 / 4\pi\hbar |\mathbf{v}_e|$ , where we have written  $e_i = Z_i e$ .

The size of this parameter is estimated by its thermal average, which we denote by an overline. We use the simple Maxwell-Boltzmann distribution to estimate this average. For this classical distribution, the thermal average of  $1/v_e^2$  is precisely  $m_e/T_e$ , and so

$$\overline{\eta_{ei}^2} \simeq Z_i^2 \left( \frac{e^2}{4\pi\hbar} \right)^2 \frac{m_e}{T_e} = Z_i^2 2\pi \eta_e^2, \quad (6.9)$$

where in the second equality we have used the previous definition (3.4) of the electron quantum Coulomb parameter  $\eta_e$  together with the definition (2.7) of the electron thermal wave length  $\lambda_e$ . Another way to write this is

$$\overline{\eta_{ei}^2} \simeq Z_i^2 \frac{2\epsilon_H}{T_e}, \quad (6.10)$$

where  $\epsilon_H \simeq 13.6$  eV previously noted in Eq. (1.11) is the binding energy of the hydrogen atom. The result (6.10) demonstrates that  $\eta_{ei}$  is quite small for the elevated temperature range that concerns us. Hence the scattering amplitude in Eq. (6.7) can be calculated in the Born approximation [15],

$$T \simeq T_B(q^2) = \hbar \frac{e e_i}{q^2}, \quad (6.11)$$

a quantity that depends only upon the square of the momentum transfer  $q^2$ , and not on the center-of-mass energy  $W$ .

In the Born approximation, the initial ion momentum  $\mathbf{p}_i$  appears only in the delta-function and phase-space density explicitly shown in Eq. (6.7), and not in the amplitude  $T_B(q^2)$ , and so the integration over this momentum variable can be carried out. If it were not for the delta-function factor, the  $\mathbf{p}_i$ -integration would simply entail

$$\begin{aligned} \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) &= \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \exp\left\{-\beta_I \left[\frac{p_i^2}{2m_i} - \mu_i\right]\right\} \\ &= \lambda_i^{-\nu} e^{\beta_I \mu_i} = n_i/g_i. \end{aligned} \quad (6.12)$$

Following the convention exhibited in Eq. (2.5), the species index  $i$  for ions implicitly includes spin degrees of freedom, and so the integration over a single  $f_i(\mathbf{p}_i)$  produces  $n_i/g_i$ . The delta-function in Eq. (6.7) removes one of the components of the  $p_i$ -integration, which is equivalent to supplying an extra factor of  $\lambda_i$  and retaining a Maxwell-Boltzmann factor corresponding to the component of the momentum  $\mathbf{p}_i$  along the direction of  $\mathbf{q}$ . Hence

$$\begin{aligned} \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) (2\pi\hbar) \delta\left(\frac{\mathbf{p}_i \cdot \mathbf{q}}{m_i} - \frac{\bar{\mathbf{p}} \cdot \mathbf{q}}{m_e} - \frac{q^2}{2m_i}\right) \\ = \frac{1}{q} \frac{n_i}{g_i} \lambda_i m_i \exp\left\{-\frac{\beta_I}{2m_i q^2} \left(\frac{m_i}{m_e} \bar{\mathbf{p}} \cdot \mathbf{q} + \frac{q^2}{2}\right)^2\right\}. \end{aligned} \quad (6.13)$$

We shall often denote the magnitude of the momentum transfer by  $q = |\mathbf{q}|$ , as we have done here. We now change the remaining two integration variables  $\mathbf{p}'_e$  and  $\mathbf{p}_e$  in Eq. (6.7) to the variables  $\bar{\mathbf{p}}$  and  $\mathbf{q}$  defined in Eq's. (6.5) and (6.6), a change that has a unit Jacobian. Since the electrons are described by the Fermi-Dirac distribution (2.2), the Pauli blocking term in Eq. (6.7) can be written as

$$\begin{aligned} 1 - f_e(\bar{\mathbf{p}} + \mathbf{q}/2) \\ = e^{-\beta_e \mu_e} \exp\left\{\frac{\beta_e}{2m_e} \left(\bar{\mathbf{p}} + \frac{1}{2}\mathbf{q}\right)^2\right\} f_e(\bar{\mathbf{p}} + \mathbf{q}/2). \end{aligned} \quad (6.14)$$

Using these results, and neglecting terms involving the very small ratios  $m_e/m_i$  and  $\beta_e m_e/\beta_I m_i$ , we find that

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^>}{\partial t} &= 2 \sum_i \frac{n_i}{g_i} \int \frac{d^\nu \bar{\mathbf{p}}}{(2\pi\hbar)^\nu} \frac{d^\nu q}{(2\pi\hbar)^\nu} |T_B(q^2)|^2 \frac{m_i \lambda_i}{m_e} e^{-\beta_e \mu_e} \\ &\quad f_e(\bar{\mathbf{p}} - \mathbf{q}/2) f_e(\bar{\mathbf{p}} + \mathbf{q}/2) \exp\left\{+\frac{\beta_e}{2m_e} \left[\bar{p}_\perp^2 + \frac{1}{4} q^2\right]\right\} \\ &\quad \bar{\mathbf{p}} \cdot \hat{\mathbf{q}} \exp\left\{-\frac{\beta_I}{2m_e} \frac{m_i}{m_e} \left[\bar{\mathbf{p}} \cdot \hat{\mathbf{q}} + \frac{m_e}{2m_i} \left(1 - \frac{\beta_e}{\beta_I}\right) q\right]^2\right\}, \end{aligned} \quad (6.15)$$

where  $\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$ , the variable  $\bar{\mathbf{p}}_\perp$  in the first exponent is the component of  $\bar{\mathbf{p}}$  orthogonal to the momentum transfer  $\mathbf{q}$ , so that  $\bar{\mathbf{p}} = \bar{\mathbf{p}}_\perp + (\bar{\mathbf{p}} \cdot \hat{\mathbf{q}}) \hat{\mathbf{q}}$  with  $\bar{\mathbf{p}}_\perp \cdot \mathbf{q} = 0$ .

We can simplify the rate (6.15) by further exploiting the consequences of the very small ratio  $m_e/m_i$ . In the order of magnitude estimates that follow, we will use the symbol  $\beta$  to designate the inverse temperatures of either the electrons or the ions. This is possible because the temperature disparity is not very severe. We now see that the thermal distribution functions in the second line of Eq. (6.15) restrict the size of the momenta to be of the

order

$$\bar{p}_\perp^2 \sim \frac{m_e}{\beta} \quad \text{and} \quad q^2 \sim \frac{m_e}{\beta}. \quad (6.16)$$

For the longitudinal component of the electron momentum, the form of the exponential in the last line of Eq. (6.15) motivates the change of variables to

$$\bar{p}'_\parallel \equiv \bar{\mathbf{p}} \cdot \hat{\mathbf{q}} + \frac{m_e}{2m_i} \left(1 - \frac{\beta_e}{\beta_I}\right) q. \quad (6.17)$$

Under this change of variables, the last line in Eq. (6.15) becomes

$$\left[ \bar{p}'_{\parallel} - \frac{m_e}{2m_i} \left( 1 - \frac{\beta_e}{\beta_i} \right) q \right] \exp \left\{ -\frac{\beta_I}{2m_e} \frac{m_i}{m_e} \bar{p}'_{\parallel}{}^2 \right\}, \quad (6.18)$$

a term whose exponent restricts the size of the longitudinal component to be

$$|\bar{p}'_{\parallel}| \sim \sqrt{\frac{m_e^2}{\beta m_i}} \sim \sqrt{\frac{m_e}{m_i}} \bar{p}_{\perp} \sim \sqrt{\frac{m_e}{m_i}} q. \quad (6.19)$$

This means that the second term in square brackets at the start of expression (6.18), the term  $(m_e/m_i)q \sim \sqrt{m_e^3/\beta m_i^2}$ , is a factor  $\sqrt{m_e/m_i}$  smaller than the first term  $\bar{p}'_{\parallel}$ . However, as we shall find, the first term integrates identically to zero, leaving the ostensibly smaller second term as the leading order contribution. To see this, we first note that the electron distributions  $f_e(\bar{\mathbf{p}} \mp \mathbf{q}/2)$  in Eq. (6.15) are functions of the dimensionless variables

$$\beta_e E_e(\bar{\mathbf{p}} \mp \mathbf{q}/2) = \frac{\beta_e}{2m_e} (\bar{\mathbf{p}} \mp \mathbf{q}/2)^2. \quad (6.20)$$

Here, we must express the old variable  $\bar{\mathbf{p}}$  in terms of the new variable

$$\bar{\mathbf{p}}' \equiv \bar{\mathbf{p}}_{\perp} + \bar{p}'_{\parallel} \hat{\mathbf{q}}, \quad (6.21)$$

or in terms of the vectors (6.5) and (6.6),

$$\bar{\mathbf{p}}' = \bar{\mathbf{p}} + \frac{m_e}{2m_i} \left( 1 - \frac{\beta_e}{\beta_i} \right) \mathbf{q}. \quad (6.22)$$

Then from Eq. (6.22), we see that replacing the old variable  $\bar{\mathbf{p}}$  in Eq. (6.20) by the new variable  $\bar{\mathbf{p}}'$  incurs relative error of order

$$(m_e/m_i) \bar{\mathbf{p}}' \cdot \mathbf{q} (1/\bar{p}^2) \sim (m_e/m_i)^{3/2}, \quad (6.23)$$

an error beyond the leading term that we retain. That is to say, we can simply replace

$$f_e(\bar{\mathbf{p}} - \mathbf{q}/2) f_e(\bar{\mathbf{p}} + \mathbf{q}/2) \rightarrow f_e(\bar{\mathbf{p}}' - \mathbf{q}/2) f_e(\bar{\mathbf{p}}' + \mathbf{q}/2). \quad (6.24)$$

This product is explicitly even in  $\mathbf{q}$ , as are the remaining terms in the integrand, and consequently, the odd term  $\bar{p}'_{\parallel}$  in the prefactor of (6.18) integrates to zero. The energy rate (6.15) now reduces to

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^>}{\partial t} = & -2 \sum_i \frac{n_i}{\mathbf{g}_i} \int \frac{d^{\nu} \bar{\mathbf{p}}'}{(2\pi\hbar)^{\nu}} \frac{d^{\nu} q}{(2\pi\hbar)^{\nu}} |T_B(q^2)|^2 \frac{m_i \lambda_i}{m_e} e^{-\beta_e \mu_e} \\ & f_e(\bar{\mathbf{p}}' - \mathbf{q}/2) f_e(\bar{\mathbf{p}}' + \mathbf{q}/2) \exp \left\{ +\frac{\beta_e}{2m_e} \left[ \bar{p}_{\perp}^2 + \frac{1}{4} q^2 \right] \right\} \\ & \frac{m_e}{2m_i} \left( 1 - \frac{\beta_e}{\beta_i} \right) q \exp \left\{ -\frac{\beta_I}{2m_e} \frac{m_i}{m_e} \bar{p}'_{\parallel}{}^2 \right\}. \end{aligned} \quad (6.25)$$

The integral over the momentum  $\bar{\mathbf{p}}' = \bar{\mathbf{p}}_{\perp} + \bar{p}'_{\parallel} \hat{\mathbf{q}}$  contains  $\nu - 1$  integrals from  $\bar{\mathbf{p}}_{\perp}$  and one integral from  $\bar{p}'_{\parallel}$ .

Now that the leading contribution has been extracted, we can make further reductions by omitting several more terms in  $m_e/m_i$ . In particular, we may now neglect the longitudinal part  $\bar{p}'_{\parallel} = \bar{\mathbf{p}}' \cdot \hat{\mathbf{q}}$  relative to  $q = |\mathbf{q}|$  in the electron distribution functions  $f_e(\bar{\mathbf{p}}' \mp \mathbf{q})$ , which then become functions only of  $(\bar{\mathbf{p}}_{\perp} \mp \mathbf{q}/2)^2$ . In fact, since  $\bar{\mathbf{p}}_{\perp} \cdot \mathbf{q} = 0$ , both electron distribution functions have the same argument,

$$\beta_e E_e(\bar{\mathbf{p}}_{\perp} \pm \mathbf{q}/2) = \frac{\beta_e}{2m_e} \left( \bar{p}_{\perp}^2 + \frac{1}{4} q^2 \right), \quad (6.26)$$

and their product becomes a simple square:  $f_e(\bar{\mathbf{p}}' - \mathbf{q}/2) f_e(\bar{\mathbf{p}}' + \mathbf{q}/2) = [f_e(\bar{\mathbf{p}}_{\perp} + \mathbf{q}/2)]^2$ . The longitudinal component  $\bar{p}'_{\parallel}$  now appears only in the final factor of the integrand in Eq. (6.25), and we may therefore explicitly perform the integration over this part of the momentum,

$$\int_{-\infty}^{\infty} \frac{d\bar{p}'_{\parallel}}{2\pi\hbar} \exp \left\{ -\frac{\beta_I}{2m_i} \left( \frac{m_i}{m_e} \right)^2 \bar{p}'_{\parallel}{}^2 \right\} = \frac{m_e}{m_i \lambda_i}, \quad (6.27)$$

where the ionic thermal wave-length  $\lambda_i$  is determined from Eq. (2.7). We can now express the rate (6.25) as

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^>}{\partial t} = & -m_e \left(1 - \frac{\beta_e}{\beta_I}\right) \sum_i \frac{n_i}{g_i m_i} \int \frac{d^{\nu-1} \bar{p}_\perp}{(2\pi\hbar)^{\nu-1}} \frac{d^\nu q}{(2\pi\hbar)^\nu} |T_B(q^2)|^2 q \\ & e^{-\beta_e \mu_e} \left[ f_e(\bar{\mathbf{p}}_\perp + \mathbf{q}/2) \right]^2 \exp \left\{ + \frac{\beta_e}{2m_e} \left[ \bar{p}_\perp^2 + \frac{1}{4} q^2 \right] \right\}. \end{aligned} \quad (6.28)$$

The integration over the momentum transfer  $\mathbf{q}$  is damped at large values of  $q = |\mathbf{q}|$ , because at such large values

$$\begin{aligned} q \rightarrow \infty : \\ e^{-\beta_e \mu_e} \left[ f_e(\bar{\mathbf{p}}_\perp + \mathbf{q}/2) \right]^2 \exp \left\{ + \frac{\beta_e}{2m_e} \left[ \bar{p}_\perp^2 + \frac{1}{4} q^2 \right] \right\} \rightarrow e^{+\beta_e \mu_e} \exp \left\{ - \frac{\beta_e}{2m_e} \left[ \bar{p}_\perp^2 + \frac{1}{4} q^2 \right] \right\}. \end{aligned} \quad (6.29)$$

Since the limit (6.29) constrains the integrand to small- $q$ , this further supports the use of the Born Approximation (6.11), which allows us to express the rate (6.28) as

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^>}{\partial t} = & -\beta_e e^2 m_e \hbar^2 \omega_I^2 (T_e - T_I) \int \frac{d^{\nu-1} \bar{p}_\perp}{(2\pi\hbar)^{\nu-1}} \frac{d^\nu q}{(2\pi\hbar)^\nu} \frac{1}{q^3} \\ & e^{-\beta_e \mu_e} \left[ f_e(\bar{\mathbf{p}}_\perp + \mathbf{q}/2) \right]^2 \exp \left\{ + \frac{\beta_e}{2m_e} \left[ \bar{p}_\perp^2 + \frac{1}{4} q^2 \right] \right\}, \end{aligned} \quad (6.30)$$

where  $\omega_I^2$  is the sum over all the ionic species of the squared plasma frequencies,

$$\omega_I^2 = \sum_{\text{ion species}} \omega_i^2 = \sum_i \frac{e_i^2 n_i}{m_i}. \quad (6.31)$$

Now that the clutter has abated, we can more easily study the nature of the parameters that enter into the  $\nu > 3$  contribution of the energy exchange rate. Dividing the time derivative of the electron energy density by the the electron specific heat  $3n_e/2$  gives the rate of the electron temperature change already noted in (5.7), namely

$$\frac{\partial T_e}{\partial t} = -\gamma_{eI} (T_e - T_I). \quad (6.32)$$

The integral of each momentum, with the normalizing denominator  $2\pi\hbar$ , gives a pure number times a factor of  $1/\lambda_e$ . Since  $n_e \sim \lambda_e^{-\nu}$ , and each factor of the momentum transfer  $|\mathbf{q}|$  will produce a factor of  $\hbar/\lambda_e$ , we conclude from Eq. (6.30) that

$$\begin{aligned} \gamma_{eI} & \sim \beta_e e^2 m_e \hbar^2 \omega_I^2 \lambda_e^{1-\nu} (\lambda_e/\hbar)^3 \\ & \sim \left( \frac{e^2}{\lambda_e^{\nu-2}} \frac{1}{T_e} \right) \left( \frac{\hbar \omega_I}{T_e} \right) \omega_I, \end{aligned} \quad (6.33)$$

where in the second line we have made use of

$$\lambda_e^2 \sim \hbar^2 \beta_e / m_e \quad \text{and} \quad \beta_e = 1/T_e. \quad (6.34)$$

In a  $\nu$ -dimensional space, the energy between two electrons a distance  $\lambda_e$  apart is, up to a constant, given by

$e^2/\lambda_e^{\nu-2}$ . Hence the first factor in parenthesis in the last line above is dimensionless. Since  $\hbar \omega_I$  is an energy, the second factor is also dimensionless. Thus the overall dimension of  $\gamma_{eI}$  is that of the final factor  $\omega_I$ , the correct dimension of an inverse time or rate. Although a factor of  $\hbar$  appears here, it is canceled by the single factor of  $\hbar$  that appears in  $1/\lambda_e^{\nu-2}$  in the  $\nu \rightarrow 3$  limit, and so in this limit the rate is a classical quantity. However, as we shall see, the dimensional continuation method that we use produces logarithms, and a logarithm of  $\hbar$  will appear in the final result. Finally, we should note that the rate involves the first power of the ion density, a power that does not depend upon the spatial dimensionality  $\nu$ .

The rate (6.30) for  $\nu > 3$  diverges when  $\nu \rightarrow 3^+$ , a divergence that is canceled by the  $\nu \rightarrow 3^-$  limit of the rate for  $\nu < 3$  that we compute in the next section. This latter rate involves purely classical dynamics. Thus it entails a wave-number  $\mathbf{k}$  that comes from the Fourier transform of a potential which is the analog of the quantum momentum transfer  $\mathbf{q}$ , but with  $\mathbf{q} = \hbar \mathbf{k}$ . With this replacement, the electron distributions would become  $f_e(\bar{\mathbf{p}}' + \hbar \mathbf{k}/2)$ , but since only classical quantities appear in the forthcoming  $\nu < 3$  contribution, in this part the electron distributions must appear only as  $f_e(\bar{\mathbf{p}}')$ . Thus, to separate out a part of the  $\nu > 3$  Boltzmann expression for the rate that will combine in a simple fashion with the  $\nu < 3$  contribution that we shall soon examine, we construct this part by making the replacement

$$\begin{aligned} & \left[ f_e(\bar{\mathbf{p}}_\perp + \mathbf{q}/2) \right]^2 \exp \left\{ + \frac{\beta_e}{8m_e} q^2 \right\} \\ \rightarrow & \left[ f_e(\bar{\mathbf{p}}_\perp) \right]^2 \exp \left\{ - \frac{\beta_e}{8m_e} q^2 \right\}, \end{aligned} \quad (6.35)$$

which exhibits the needed large  $q^2$  damping given in the limit (6.29). Accordingly, we decompose the rate of energy transfer into a potentially singular part and a regular part,

$$\frac{\partial \mathcal{E}_{eI}^{>}}{\partial t} = \frac{\partial \mathcal{E}_{eI}^{>s}}{\partial t} + \frac{\partial \mathcal{E}_{eI}^{>r}}{\partial t}, \quad (6.36)$$


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$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^{>s}}{\partial t} = & -\beta_e e^2 m_e \hbar^2 \omega_I^2 (T_e - T_I) \int \frac{d^{\nu-1} \bar{p}_\perp}{(2\pi\hbar)^{\nu-1}} \frac{d^\nu q}{(2\pi\hbar)^\nu} \frac{1}{q^3} \exp\left\{-\frac{\beta_e}{8m_e} q^2\right\} \\ & e^{-\beta_e \mu_e} \left[f_e(\bar{\mathbf{p}}_\perp)\right]^2 \exp\left\{+\frac{\beta_e}{2m_e} \bar{p}_\perp^2\right\}, \end{aligned} \quad (6.37)$$

and

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^{>r}}{\partial t} = & -\beta_e e^2 m_e \hbar^2 \omega_I^2 (T_e - T_I) \int \frac{d^2 \bar{p}_\perp}{(2\pi\hbar)^2} \frac{d^3 q}{(2\pi\hbar)^3} \frac{1}{q^3} e^{-\beta_e \mu_e} \\ & \left( \left[f_e(\bar{\mathbf{p}}_\perp + \mathbf{q}/2)\right]^2 \exp\left\{+\frac{\beta_e}{2m_e} \left(\bar{p}_\perp^2 + \frac{1}{4} q^2\right)\right\} - \left[f_e(\bar{\mathbf{p}}_\perp)\right]^2 \exp\left\{+\frac{\beta_e}{2m_e} \left(\bar{p}_\perp^2 - \frac{1}{4} q^2\right)\right\} \right). \end{aligned} \quad (6.38)$$

Here in the regular part, we have taken the  $\nu \rightarrow 3$  limit since there is no impediment in so doing.

The singular part (6.37) may be simplified by performing the  $q$ -integration. Passing to hyper-spherical coordinates gives

$$\begin{aligned} \int \frac{d^\nu q}{(2\pi\hbar)^\nu} \frac{1}{q^3} \exp\left\{-\frac{\beta_e}{8m_e} q^2\right\} &= \frac{\Omega_{\nu-1}}{(2\pi\hbar)^\nu} \int_0^\infty \frac{dq}{q} q^{\nu-3} \exp\left\{-\frac{\beta_e}{8m_e} q^2\right\} \\ &= \frac{\Omega_{\nu-1}}{(2\pi\hbar)^\nu} \frac{1}{2} \int_0^\infty \frac{dx}{x} \left(\frac{8m_e}{\beta_e} x\right)^{(\nu-3)/2} e^{-x} \\ &= \frac{\Omega_{\nu-1}}{(2\pi\hbar)^3} \frac{1}{2} \left(\frac{\pi \lambda_e^2}{4}\right)^{(3-\nu)/2} \Gamma\left(\frac{\nu-3}{2}\right), \end{aligned} \quad (6.39)$$

where  $\Omega_{\nu-1}$  is the area of a unit  $(\nu-1)$ -sphere embedded in a  $\nu$ -dimensional space. In the second line above we have made an obvious change to a dimensionless integration variable  $x$ , and in the last line we have identified the resulting integral with a standard representation of the gamma function. We thus have

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^{>s}}{\partial t} = & -\frac{\beta_e e^2 m_e}{2\hbar} \omega_I^2 \frac{\Omega_{\nu-1}}{(2\pi)^3} \left(\frac{\pi \lambda_e^2}{4}\right)^{(3-\nu)/2} \Gamma\left(\frac{\nu-3}{2}\right) (T_e - T_I) \\ & \int \frac{d^{\nu-1} \bar{p}_\perp}{(2\pi\hbar)^{\nu-1}} \left[f_e(\bar{\mathbf{p}}_\perp)\right]^2 \exp\left\{\beta_e \left[\frac{\bar{p}_\perp^2}{2m_e} - \mu_e\right]\right\}. \end{aligned} \quad (6.40)$$

As we shall see, the  $\nu < 3$  contribution calculated in the next section contains the same integral over the transverse components  $\bar{\mathbf{p}}_\perp$ , so it will be convenient to perform this integral when we add these terms together.

The regular part of the energy exchange rate may also be simplified since the integral over the electron distribution functions can be performed when  $\nu = 3$ . Namely, we pass to polar coordinates, with the angular integration simply



producing a factor of  $2\pi$ , to obtain

$$\begin{aligned}
& \int \frac{d^2 \bar{p}_\perp}{(2\pi\hbar)^2} \left[ f_e(\bar{\mathbf{p}}_\perp) \right]^2 \exp \left\{ \beta_e \left[ \frac{\bar{p}_\perp^2}{2m_e} - \mu_e \right] \right\} \\
&= \frac{m_e}{2\pi\beta_e \hbar^2} \int_0^\infty d \left( \frac{\beta_e p^2}{2m_e} \right) \frac{\exp \left\{ +\beta_e \left[ \frac{p^2}{2m_e} - \mu_e \right] \right\}}{\left[ \exp \left\{ +\beta_e \left[ \frac{p^2}{2m_e} - \mu_e \right] \right\} + 1 \right]^2} \\
&= \frac{1}{\lambda_e^2} \frac{1}{\exp \{-\beta_e \mu_e\} + 1}.
\end{aligned} \tag{6.41}$$

Hence, making the replacement

$$-\beta_e \mu_e \rightarrow \beta_e \left[ \frac{q^2}{8m_e} - \mu_e \right] \tag{6.42}$$

for the first term, we have

$$\begin{aligned}
\frac{\partial \mathcal{E}_{eI}^{>R}}{\partial t} &= -\frac{\beta_e e^2 m_e}{2\hbar} \frac{\omega_I^2}{\pi^2} \frac{1}{\lambda_e^2} (T_e - T_I) \int_0^\infty \frac{dq}{q} \\
&\quad \left\{ \frac{1}{\exp \left\{ \beta_e \left[ \frac{q^2}{8m_e} - \mu_e \right] \right\} + 1} - \frac{\exp \left\{ -\beta_e \frac{q^2}{8m_e} \right\}}{\exp \{-\beta_e \mu_e\} + 1} \right\} \\
&= -\frac{\beta_e e^2 m_e}{2\hbar} \frac{\omega_I^2}{\pi^2} \frac{1}{\lambda_e^2} (T_e - T_I) \frac{1}{2} \int_0^\infty dx \ln x \\
&\quad \left\{ \frac{\exp \{x - \beta_e \mu_e\}}{[\exp \{x - \beta_e \mu_e\} + 1]^2} - \frac{\exp \{-x\}}{\exp \{-\beta_e \mu_e\} + 1} \right\},
\end{aligned} \tag{6.43}$$

where the last line follows from a trivial change of integration variables and a partial integration.

## VII. LENARD-BALESCU EQUATION: LONG-DISTANCE PHYSICS

We turn now to calculate the leading order long-distance physics by working in spatial dimensions  $\nu < 3$ . This is done by employing the Lenard-Balescu equation with Fermi-Dirac statistics for the electrons and Maxwell-Boltzmann statistics for the heavy ions. For the spatially homogeneous system that we work with, the Lenard-Balescu equation with the appropriate Pauli-blocking reads

$$\begin{aligned}
\frac{\partial f_e(\mathbf{p}_e)}{\partial t} &= -\frac{\partial}{\partial \mathbf{p}_e} \cdot \sum_i \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \mathbf{k} \left| \frac{e e_i}{k^2 \epsilon(k, \mathbf{k} \cdot \mathbf{v}_i)} \right|^2 \pi \delta(\mathbf{k} \cdot \mathbf{v}_e - \mathbf{k} \cdot \mathbf{v}_i) \\
&\quad \left\{ \mathbf{k} \cdot \frac{\partial f_i(\mathbf{p}_i)}{\partial \mathbf{p}_i} f_e(\mathbf{p}_e) [1 - f_e(\mathbf{p}_e)] - f_i(\mathbf{p}_i) \mathbf{k} \cdot \frac{\partial f_e(\mathbf{p}_e)}{\partial \mathbf{p}_e} \right\},
\end{aligned} \tag{7.1}$$

where the gradient  $\partial/\partial \mathbf{p}_e$  acts on everything to its right, and  $\epsilon(k, \omega)$  is the classical dielectric function for the plasma discussed in Appendix A. As was shown in Appendix C of Ref. [1], the usual non-degenerate Lenard-Balescu equation is a formal limit of the Boltzmann equation. The same methods that were employed there may be used to show that Eq. (7.1) is the corresponding long-distance equation when the electrons are degenerate and described by Fermi-Dirac statistics. If the electrons and ions are in equilibrium with themselves at temperatures  $T_e$  and  $T_i$ , respectively, then their distribution functions  $f_e$  and  $f_i$  are given by Eq's. (2.2) and (2.1), in which case the terms in curly braces can be written as

$$\left( \beta_e \mathbf{k} \cdot \mathbf{v}_e - \beta_i \mathbf{k} \cdot \mathbf{v}_i \right) f_i(\mathbf{p}_i) \left[ f_e(\mathbf{p}_e) \right]^2 \exp \left\{ \beta_e \left[ \frac{p_e^2}{2m_e} - \mu_e \right] \right\}. \tag{7.2}$$

Because the delta-function equates  $\mathbf{k} \cdot \mathbf{v}_e$  with  $\mathbf{k} \cdot \mathbf{v}_i$ , the factor (7.2) and with it the right-hand-side of Eq. (7.1) vanishes when the electrons and ions are in thermal equilibrium with a common temperature  $T$ , the electrons being described by a Fermi-Dirac distribution and the ions by a Maxwell-Boltzmann distribution. This confirms the validity of Eq. (7.1), the Lenard-Balescu equation with Pauli-blocking for degenerate electrons.

Using (7.2) with the ions at the same inverse temperature  $\beta_I = 1/T_I$ , and upon integrating the total derivative  $\partial/\partial \mathbf{p}_e$  by parts, we can express the energy exchange rate as

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^<}{\partial t} = & 2 \sum_i \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \mathbf{k} \cdot \mathbf{v}_e \left| \frac{e e_i}{k^2 \epsilon(k, \mathbf{k} \cdot \mathbf{v}_i)} \right|^2 \pi \delta(\mathbf{k} \cdot \mathbf{v}_e - \mathbf{k} \cdot \mathbf{v}_i) \\ & \left( \beta_e \mathbf{k} \cdot \mathbf{v}_e - \beta_I \mathbf{k} \cdot \mathbf{v}_i \right) f_i(\mathbf{p}_i) \left[ f_e(\mathbf{p}_e) \right]^2 \exp \left\{ \beta_e \left[ \frac{p_e^2}{2m_e} - \mu_e \right] \right\}. \end{aligned} \quad (7.3)$$

We have placed a “less-than” superscript on the left-hand side of (7.3) to remind ourselves that the calculation is performed in  $\nu < 3$  using the Lenard-Balescu equation. The distribution functions constrain the velocities of the ions and the electrons to be of the order  $v_i \sim \sqrt{T_I/m_i}$  and  $v_e \sim \sqrt{T_e/m_e}$ , respectively. Since an ion mass is so much greater than that of an electron, and the temperature disparity is never excessively large for cases of interest, the ions move much slower than the electrons,

$$v_i \ll v_e. \quad (7.4)$$

This restriction also follows from the previous condition (6.8), the condition that  $\beta_e m_e \ll \beta_I m_i$ . To compute the rate (7.3), we first decompose the electron momentum into perpendicular and longitudinal components relative to the direction specified by  $\hat{\mathbf{k}}$ , so that  $\mathbf{p}_e = \mathbf{p}_\perp + p_\parallel \hat{\mathbf{k}}$  with  $\hat{\mathbf{k}} \cdot \mathbf{p}_\perp = 0$  and  $p_\parallel = \hat{\mathbf{k}} \cdot \mathbf{p}_e = m_e \hat{\mathbf{k}} \cdot \mathbf{v}_e$ . The delta

function in Eq. (7.3) can be used to remove the parallel component  $p_\parallel$  of the electron momenta integration. Since  $dp_\parallel/(2\pi\hbar) = (m_e/2\pi\hbar k) d(\mathbf{k} \cdot \mathbf{v}_e)$ , the use of this delta function produces a factor  $(m_e/2\pi\hbar k)$  and makes the replacement  $p_\parallel \rightarrow m_e \hat{\mathbf{k}} \cdot \mathbf{v}_i$ . In view of the limit (7.4), this replacement makes  $p_\parallel$  much smaller than the magnitude of the perpendicular components of the electron momenta  $p_\perp$ . Hence, we can simply replace  $\mathbf{p} \rightarrow \mathbf{p}_\perp$  in the remainder of the integrand. We shall also find it convenient (the heart hath its reasons), to insert a factor of unity in the form

$$1 = \int_{-\infty}^{+\infty} dv \delta(v - \hat{\mathbf{k}} \cdot \mathbf{v}_i), \quad (7.5)$$

which allows us to express Eq. (7.3) as

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^<}{\partial t} = & -2 \beta_e \frac{e^2 m_e}{2\pi\hbar} (T_e - T_I) \int \frac{d^{\nu-1} p_\perp}{(2\pi\hbar)^{\nu-1}} \left[ f_e(\mathbf{p}_\perp) \right]^2 \exp \left\{ \beta_e \left[ \frac{p_\perp^2}{2m_e} - \mu_e \right] \right\} \\ & \int \frac{d^\nu k}{(2\pi)^\nu} k \int_{-\infty}^{+\infty} dv \frac{\pi v^2}{[k^2 \epsilon(k, vk)]^2} \sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) \delta(\hat{\mathbf{k}} \cdot \mathbf{v}_i - v). \end{aligned} \quad (7.6)$$

This integral can be further simplified by taking advantage of the analytic properties of the dielectric function  $\epsilon(k, \omega)$ , discussed in some detail in Appendix A. Repeating Eq. (A16) here for convenience, we see that a considerable portion of the integral simplifies because

$$\begin{aligned} & \frac{\pi v}{[k^2 \epsilon(k, vk)]^2} \sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) \delta(\hat{\mathbf{k}} \cdot \mathbf{v}_i - v) \\ & = -\frac{1}{2i} \left\{ \frac{1}{k^2 + \kappa_e^2 + F_I(v)} - \frac{1}{k^2 + \kappa_e^2 + F_I(-v)} \right\}. \end{aligned} \quad (7.7)$$

This result is the (unemotional) reason that the factor of unity in the form displayed in Eq. (7.5) was inserted in the integrand. Here  $\kappa_e$  is the electronic contribution to the Debye wave number, including the effects of Fermi-Dirac statistics, as expressed by Eq. (A6), while  $F_I$  is a complex-valued function defined by Eq. (A9). It is important to realize that the simplification (7.7) only occurs when the ion species are summed over. The function  $F_I(z)$  is analytic over the upper half of the complex  $z$ -plane, and has the asymptotic behavior

$$|z| \rightarrow \infty : \quad F_I(z) \rightarrow -\frac{\omega_I^2}{z^2}, \quad (7.8)$$

where  $\omega_I$  is the total ionic plasma frequency defined above in Eq. (6.31). Since an explicit odd factor of  $v$  appears in the integrand, we can write the resulting integral over  $v$  in Eq's. (7.6) in the form

$$-\int_{-\infty}^{+\infty} dv \frac{v}{2i} \left\{ \frac{1}{k^2 + \kappa_e^2 + F_I(v)} - \frac{1}{k^2 + \kappa_e^2 + F_I(-v)} \right\} = \lim_{V \rightarrow \infty} i \int_{-V}^{+V} dv \frac{v}{k^2 + \kappa_e^2 + F_I(v)}. \quad (7.9)$$

The delta-function in Eq. (7.7) removes the longitudinal components of the ionic momenta, leaving a Maxwell-Boltzmann factor involving the velocity  $v$ . Hence the left-hand-side of the integrand in Eq. (7.9) is damped in a Gaussian fashion for large  $|v|$ . This rapid damping results from a cancellation between the terms with  $F_I(v)$  and  $F_I(-v)$  that only happens when the two velocities are exactly the negative of one another. Hence, when we simplify the integrand by taking advantage of the odd prefactor as was done on the right-hand side of Eq. (7.9), we must integrate between the exact same negative and positive limits, between  $-V$  and  $+V$ , and only afterward take the limit  $V \rightarrow \infty$ . Since  $F_I(z)$  is analytic in the upper-half  $z$ -plane, the integral (7.9) may be evaluated by contour integral techniques. Let  $C_V$  be a semicircle of radius  $V$  centered at the origin of the complex  $z$ -plane, with an orientation that starts at  $+V$  and ends at  $-V$ . We can traverse a closed circuit by moving from  $-V$  to  $+V$  along the real axis, with the circuit completed back to  $-V$  by traversing  $C_V$ . The contour integral around this closed circuit vanishes since it contains no interior singularities,

$$\begin{aligned} 0 &= \oint dz \frac{z}{k^2 + \kappa_e^2 + F_I(z)} \\ &= \int_{-V}^{+V} dv \frac{v}{k^2 + \kappa_e^2 + F_I(v)} + \int_{C_V} dz \frac{z}{k^2 + \kappa_e^2 + F_I(z)}. \end{aligned} \quad (7.10)$$

Hence the integral (7.9) is equal to the negative of the integral over the semicircle  $C_V$  starting at  $+V$  and ending at  $-V$ . We can now take the limit  $V \rightarrow \infty$  and use the asymptotic form (7.8) for  $F_I$  along  $C_V$ . Since

$$\int_{C_V} dz z = i V^2 \int_0^\pi d\theta e^{2i\theta} = 0, \quad (7.11)$$

the leading term of the expansion of the denominator in Eq. (7.9) yields a vanishing contribution, and the only non-vanishing term is given by first-order term with  $F_I(z)$  replaced with its asymptotic form (7.8):

$$\begin{aligned} \lim_{V \rightarrow \infty} i \int_{-V}^{+V} dv \frac{v}{k^2 + \kappa_e^2 + F_I(v)} &= i \frac{1}{[k^2 + \kappa_e^2]^2} \lim_{V \rightarrow \infty} \int_{C_V} dz z \left[ -\frac{\omega_I^2}{z^2} \right] \\ &= \frac{\pi}{[k^2 + \kappa_e^2]^2} \omega_I^2. \end{aligned} \quad (7.12)$$

Upon passing to hyper-spherical coordinates to perform the  $k$ -integration, we now arrive at

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^<}{\partial t} &= -\frac{\beta_e e^2 m_e}{\hbar} (T_e - T_I) \int \frac{d^{\nu-1} p_\perp}{(2\pi\hbar)^{\nu-1}} \left[ f_e(\mathbf{p}_\perp) \right]^2 \exp \left\{ \beta_e \left[ \frac{p_\perp^2}{2m_e} - \mu_e \right] \right\} \\ &\quad \omega_I^2 \frac{\Omega_{\nu-1}}{(2\pi)^\nu} \int_0^\infty k^{\nu-1} dk k \frac{1}{[k^2 + \kappa_e^2]^2}. \end{aligned} \quad (7.13)$$

Changing variables by  $k = t^{1/2} \kappa_e$  places the  $k$ -integration in the form of a standard representation of the Euler Beta function [16], and so we have

$$\int_0^\infty dk \frac{k^\nu}{[k^2 + \kappa_e^2]^2} = \frac{1}{2} \kappa_e^{\nu-3} \frac{\Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{3-\nu}{2}\right)}{\Gamma(2)}. \quad (7.14)$$

Finally, we are now able to express the  $\nu < 3$  form of the electron-ion energy exchange as

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^<}{\partial t} &= -\frac{\beta_e e^2 m_e}{2\hbar} \omega_I^2 \frac{\Omega_{\nu-1}}{(2\pi)^3} \left( \frac{\kappa_e}{2\pi} \right)^{\nu-3} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{3-\nu}{2}\right) (T_e - T_I) \\ &\quad \int \frac{d^{\nu-1} p_\perp}{(2\pi\hbar)^{\nu-1}} \left[ f_e(\mathbf{p}_\perp) \right]^2 \exp \left\{ \beta_e \left[ \frac{p_\perp^2}{2m_e} - \mu_e \right] \right\}. \end{aligned} \quad (7.15)$$

### VIII. ADDING THE RATES

The sum of the singular part (6.40) for the  $\nu > 3$  contribution to the electron-ion energy exchange rate and the  $\nu < 3$  part (7.15) that we have just computed is

$$\begin{aligned} \frac{\partial \mathcal{E}_{eI}^{>S}}{\partial t} + \frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = & -\frac{\beta_e e^2 m_e}{2\hbar} \omega_I^2 \frac{\Omega_{\nu-1}}{(2\pi)^3} (T_e - T_I) \int \frac{d^{\nu-1} p_{\perp}}{(2\pi\hbar)^{\nu-1}} \left[ f_e(\mathbf{p}_{\perp}) \right]^2 \exp\left\{ \beta_e \left[ \frac{p_{\perp}^2}{2m_e} - \mu_e \right] \right\} \\ & \left( \frac{\pi \lambda_e^2}{4} \right)^{(3-\nu)/2} \left\{ \Gamma\left(\frac{\nu-3}{2}\right) + \left( \frac{\kappa_e^2 \lambda_e^2}{16\pi} \right)^{(\nu-3)/2} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{3-\nu}{2}\right) \right\}. \end{aligned} \quad (8.1)$$

As must be the case, the expression in the final curly braces above is finite in the  $\nu \rightarrow 3$  limit. To extract this limit, we use

$$\begin{aligned} \nu \rightarrow 3 : \\ \Gamma\left(\frac{\nu-3}{2}\right) \rightarrow \frac{2}{\nu-3} - \gamma, \quad \Gamma\left(\frac{3-\nu}{2}\right) \rightarrow \frac{2}{3-\nu} - \gamma, \quad \Gamma\left(\frac{\nu+1}{2}\right) \rightarrow 1 - (1-\gamma) \frac{3-\nu}{2} \end{aligned} \quad (8.2)$$

to evaluate the  $\nu \rightarrow 3$  limit of the last line in Eq. (8.1):

$$\left[ \frac{2}{\nu-3} - \gamma \right] + \left( \frac{\kappa_e^2 \lambda_e^2}{16\pi} \right)^{(\nu-3)/2} \left[ \frac{2}{3-\nu} - 1 \right] \rightarrow \ln\left\{ \frac{16\pi}{\kappa_e^2 \lambda_e^2} \right\} - \gamma - 1. \quad (8.3)$$

Since this last factor is finite in the  $\nu \rightarrow 3$  limit, we may now take the  $\nu \rightarrow 3$  limit of all the other quantities in Eq. (8.1). The integral over the perpendicular momenta in this limit was evaluated previously in Eq. (6.41), and so we now have

$$\frac{\partial \mathcal{E}_{eI}^{>S}}{\partial t} + \frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = -\frac{\beta_e e^2 m_e}{2\hbar} \frac{\omega_I^2}{2\pi^2} \frac{1}{\lambda_e^2} \frac{1}{\exp\{-\beta_e \mu_e\} + 1} \left[ \ln\left\{ \frac{16\pi}{\kappa_e^2 \lambda_e^2} \right\} - \gamma - 1 \right] (T_e - T_I). \quad (8.4)$$

To this we must add the remaining finite part (6.43) of the  $\nu > 3$  contribution, namely

$$\frac{\partial \mathcal{E}_{eI}^{>R}}{\partial t} = -\frac{\beta_e e^2 m_e}{2\hbar} \frac{\omega_I^2}{2\pi^2} \frac{1}{\lambda_e^2} (T_e - T_I) \int_0^\infty dx \ln x \left\{ \frac{\exp\{x - \beta_e \mu_e\}}{[\exp\{x - \beta_e \mu_e\} + 1]^2} - \frac{\exp\{-x\}}{\exp\{-\beta_e \mu_e\} + 1} \right\}. \quad (8.5)$$

Recalling that we have defined [Eq. (5.1)]

$$\frac{d\mathcal{E}_{eI}}{dt} = -\mathcal{C}_{eI} (T_e - T_I), \quad (8.6)$$

we have now calculated the rate coefficient to leading order in the plasma coupling and to all orders in the electron fugacity  $z_e = e^{\beta_e \mu_e}$ ,

$$\begin{aligned} \mathcal{C}_{eI} = & \frac{\beta_e e^2 m_e}{2\hbar} \frac{\omega_I^2}{\pi^2} \frac{1}{\lambda_e^2} \left\{ \frac{1}{\exp\{-\beta_e \mu_e\} + 1} \frac{1}{2} \left[ \ln\left\{ \frac{16\pi}{\kappa_e^2 \lambda_e^2} \right\} - \gamma - 1 \right] \right. \\ & \left. + \frac{1}{2} \int_0^\infty dx \ln x \left[ \frac{\exp\{x - \beta_e \mu_e\}}{[\exp\{x - \beta_e \mu_e\} + 1]^2} - \frac{\exp\{-x\}}{\exp\{-\beta_e \mu_e\} + 1} \right] \right\}. \end{aligned} \quad (8.7)$$

By expanding the denominators, it is easy to check that

$$\int_0^\infty dx \ln x \left[ \frac{\exp\{x - \beta_e \mu_e\}}{[\exp\{x - \beta_e \mu_e\} + 1]^2} - \frac{\exp\{-x\}}{\exp\{-\beta_e \mu_e\} + 1} \right] = \sum_{l=1}^\infty (-1)^{l+1} \ln\{l+1\} e^{(l+1)\beta_e \mu_e}, \quad (8.8)$$

which is an expansion in powers of the electron fugacity  $z_e = e^{\beta_e \mu_e}$ .

To place this result in a form that is easily compared to that of BPS [1], we use the definition (2.7) of the thermal wavelength and a slight manipulation to write

$$\begin{aligned} \mathcal{C}_{eI} = & \frac{\omega_I^2}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} \left( \frac{2\beta_e e^2}{\lambda_e^3} e^{\beta_e \mu_e} \right) \left\{ \frac{1}{\exp\{\beta_e \mu_e\} + 1} \frac{1}{2} \left[ \ln \left\{ \frac{16\pi}{\kappa_e^2 \lambda_e^2} \right\} - \gamma - 1 \right] \right. \\ & \left. + \frac{1}{2} \sum_{l=1}^{\infty} (-1)^{l+1} \ln\{l+1\} e^{l\beta_e \mu_e} \right\}. \end{aligned} \quad (8.9)$$

In the dilute limit in which Maxwell-Boltzmann statistics apply, the fugacity  $\exp\{\beta_e \mu_e\}$  is very small. The number density approximation (2.12) gives

$$\frac{2}{\lambda_e^3} e^{\beta_e \mu_e} \simeq n_e \left[ 1 + \frac{1}{2^{3/2}} e^{\beta_e \mu_e} \right], \quad (8.10)$$

and we see that keeping the first correction in the fugacity yields

$$\mathcal{C}_{eI} \simeq \frac{\omega_I^2}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} (\beta_e e^2 n_e) \left\{ \left[ 1 - \left( 1 - \frac{1}{2^{3/2}} \right) e^{\beta_e \mu_e} \right] \frac{1}{2} \left[ \ln \left\{ \frac{16\pi}{\kappa_e^2 \lambda_e^2} \right\} - \gamma - 1 \right] + \frac{1}{2} e^{\beta_e \mu_e} \ln 2 \right\}. \quad (8.11)$$

Again remembering the fugacity approximation (2.17), which we repeat here,

$$\kappa_e^2 \simeq \beta_e e^2 n_e \left[ 1 - \frac{1}{2^{3/2}} e^{\beta_e \mu_e} \right], \quad (8.12)$$

and the definitions

$$\lambda_e^2 = \frac{2\pi \hbar^2 \beta_e}{m_e} \quad \text{and} \quad \omega_e^2 = \frac{e^2 n_e}{m_e}, \quad (8.13)$$

we find that

$$\frac{16\pi}{\kappa_e^2 \lambda_e^2} \simeq \frac{8T_e^2}{\hbar^2 \omega_e^2} \left[ 1 + \frac{1}{2^{3/2}} e^{\beta_e \mu_e} \right], \quad (8.14)$$

and thus

$$\begin{aligned} \mathcal{C}_{eI} \simeq & \frac{\omega_I^2}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} (\beta_e e^2 n_e) \left\{ \left[ 1 - \left( 1 - \frac{1}{2^{3/2}} \right) e^{\beta_e \mu_e} \right] \frac{1}{2} \left[ \ln \left\{ \frac{8T_e^2}{\hbar^2 \omega_e^2} \right\} - \gamma - 1 \right] \right. \\ & \left. + e^{\beta_e \mu_e} \left[ \frac{1}{2} \ln 2 + \frac{1}{2^{5/2}} \right] \right\}. \end{aligned} \quad (8.15)$$

We may use  $e^{\beta_e \mu_e} \simeq \lambda_e^3 n_e / 2$  inside the curly braces of (8.15). It is easy to confirm that with the neglect of the fugacity corrections, this is in agreement with Eq. (12.12) of BPS [1] after that equation is corrected as mentioned in the Introduction.

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## APPENDIX A: THE DIELECTRIC FUNCTION

In Section VII, the calculation of the rate in  $\nu < 3$  using the Lenard-Balescu equation made extensive use of the plasma dielectric function and its various properties. The classical dielectric function for a collisionless plasma is discussed in Ref. [17] for example, and the form of the result that we shall use reads

$$\epsilon(k, \omega) = 1 + \sum_b \frac{e_b^2}{k^2} \int \frac{d^3 p_b}{(2\pi \hbar)^3} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_b + i\eta} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_b} f_b(\mathbf{p}_b), \quad (A1)$$

with the prescription  $\eta \rightarrow 0^+$  defining the correct retarded response. The degenerate electrons are described by the thermal Fermi-Dirac distribution (2.2), so

$$\begin{aligned} \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_e} f_e(\mathbf{p}_e) &= -\beta_e \mathbf{k} \cdot \mathbf{v}_e \frac{e^{\beta_e(E_e - \mu_e)}}{[e^{\beta_e(E_e - \mu_e)} + 1]^2} \\ &= -\beta_e \mathbf{k} \cdot \mathbf{v}_e f_e(\mathbf{p}_e) [1 - f_e(\mathbf{p}_e)]. \end{aligned} \quad (\text{A2})$$

On the other hand, the ions are described by the Maxwell-Boltzmann distribution (2.1), which is simply the large chemical potential limit  $-\beta\mu \gg 1$  of the Fermi-Dirac distribution. In this limit the Pauli blocking term is removed,  $[1 - f(\mathbf{p})] \rightarrow 1$ , and so

$$\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}_i} f_i(\mathbf{p}_i) = -\beta_i \mathbf{k} \cdot \mathbf{v}_i f_i(\mathbf{p}_i). \quad (\text{A3})$$

For the real plasma considered in the text, the ions equilibrate to a common temperature  $T_I = 1/\beta_I$ ; however, for the purposes of this Appendix, we shall take each ion species  $i$  to have an individual inverse temperature  $\beta_i$ . For degenerate electrons and Maxwell-Boltzmann ions, the dielectric function (A1) may therefore be expressed as

$$\begin{aligned} \epsilon(k, \omega) &= 1 - \frac{\beta_e e^2}{k^2} \cdot 2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{\mathbf{k} \cdot \mathbf{v}_e}{\omega - \mathbf{k} \cdot \mathbf{v}_e + i\eta} f_e(\mathbf{p}_e) [1 - f_e(\mathbf{p}_e)] \\ &\quad - \sum_i \frac{\beta_i e_i^2}{k^2} \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{\mathbf{k} \cdot \mathbf{v}_i}{\omega - \mathbf{k} \cdot \mathbf{v}_i + i\eta} f_i(\mathbf{p}_i). \end{aligned} \quad (\text{A4})$$

The factor of two in the electron contribution arises from a sum over the two spin components of the electron.

The dielectric function in the Lenard-Balescu equation has the functional form  $\epsilon(k, \mathbf{v} \cdot \mathbf{k})$ , with the speed  $|\mathbf{v}|$  much less than the electron thermal velocity. Hence, in the electron contribution to the dielectric function, the magnitude of  $\omega = \mathbf{k} \cdot \mathbf{v}$  is much less than the typical magnitude of  $\mathbf{k} \cdot \mathbf{v}_e$ , and we can use the  $\omega \rightarrow 0$  limit in which

$$\begin{aligned} &-\frac{\beta_e e^2}{k^2} \cdot 2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{\mathbf{k} \cdot \mathbf{v}_e}{\omega - \mathbf{k} \cdot \mathbf{v}_e + i\eta} f_e(\mathbf{p}_e) [1 - f_e(\mathbf{p}_e)] \\ &\quad \rightarrow \frac{\kappa_e^2}{k^2}, \end{aligned} \quad (\text{A5})$$

where

$$\kappa_e^2 = 2\beta_e e^2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} f_e(\mathbf{p}_e) [1 - f_e(\mathbf{p}_e)] \quad (\text{A6})$$

defines the electron contribution to the squared Debye wave number, including the effects of Fermi-Dirac statistics which are explicitly exhibited by the Pauli blocking factor  $[1 - f_e(\mathbf{p}_e)]$ . From the form (2.2) of the thermal Fermi-Dirac distribution  $f_e(\mathbf{p}_e)$  and the definition (2.5) of the number density, we see that

$$\kappa_e^2 = e^2 \beta_e \frac{\partial n_e}{\partial(\beta_e \mu_e)}. \quad (\text{A7})$$

Remembering the structure of the grand canonical ensemble, the derivative that appears here is the thermal

average of the fluctuations about the mean particle number. In the limit of Maxwell-Boltzmann statistics, the derivative simply reproduces the particle number density, corresponding to the fact that classical statistics have a Poisson distribution. Multiplying Eq. (A5) by  $k^2$  and using Eq. (A5) for the electron contribution, we can write

$$k^2 \epsilon(k, \omega) = k^2 + \kappa_e^2 + F_I(\omega/k), \quad (\text{A8})$$

where we have defined the function

$$F_I(v) = -\sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{\hat{\mathbf{k}} \cdot \mathbf{v}_i}{v - \hat{\mathbf{k}} \cdot \mathbf{v}_i + i\eta} f_i(\mathbf{p}_i). \quad (\text{A9})$$

This is almost same function  $F$  defined in Ref. [1], except that here we have handled the electron contribution separately.

Superficially it appears that  $F_I$  contains wave-vector dependence through the terms  $\hat{\mathbf{k}} \cdot \mathbf{v}_i$  in the integrand of (A9); however, since we are integrating over all values of  $\mathbf{v}_i$ , the wave-vector direction  $\hat{\mathbf{k}}$  cancels in  $F_I$ . As our notation suggests,  $F_I(v)$  is indeed only a function of  $v = |\mathbf{v}|$ . Furthermore, because of the  $i\eta$ -term with  $\eta > 0$  in the denominator, the function  $F_I$  is analytic in the upper complex  $v$ -plane.

In evaluating the integral (7.12) in the text, we require the large- $v$  behavior of (A9). Since the numerator of the integrand in Eq. (A9) is odd, we can expand the denominator to find the leading  $v$ -behavior

$$|v| \rightarrow \infty : \quad F_I(v) \rightarrow - \sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{(\hat{\mathbf{k}} \cdot \mathbf{v}_i)^2}{v^2} f_i(\mathbf{p}_i) = -\frac{\omega_I^2}{v^2} + \mathcal{O}(v^{-4}), \quad (\text{A10})$$

where

$$\omega_I^2 = \sum_i \frac{e_i^2 n_i}{m_i} \quad (\text{A11})$$

is the sum of the squared ion plasma frequencies. To obtain this result, the Gaussian integral in Eq. (A10) may be calculated directly, or more elegantly, one may use Eq. (A3) to replace  $(\hat{\mathbf{k}} \cdot \mathbf{v}_i) f_i$  with a derivative of  $f_i$ , after which a partial integration yields Eq. (A10).

Finally, we derive a dispersion relation that will be quite useful in evaluating Eq. (7.6) in Section VII. Applying the relation

$$\text{Im} \frac{1}{x + i\eta} = -\pi \delta(x) \quad (\text{A12})$$

for  $\eta \rightarrow 0^+$  in Eq. (A9) allows us to express the imaginary part of  $F_I$  in the form

$$\text{Im} F_I(v) = \sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) v \pi \delta(v - \hat{\mathbf{k}} \cdot \mathbf{v}_i). \quad (\text{A13})$$

From this, we can find the imaginary part of the inverse of the dielectric function:

$$\begin{aligned} \text{Im} \frac{1}{k^2 \epsilon(k, \mathbf{v} \cdot \mathbf{k})} &= -\frac{\text{Im} k^2 \epsilon(k, \mathbf{v} \cdot \mathbf{k})}{|k^2 \epsilon(k, \mathbf{v} \cdot \mathbf{k})|^2} = -\frac{\text{Im} F_I(v)}{|k^2 \epsilon(k, \mathbf{v} \cdot \mathbf{k})|^2} \\ &= -\frac{1}{|k^2 \epsilon(k, \mathbf{v} \cdot \mathbf{k})|^2} \sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) v \pi \delta(v - \hat{\mathbf{k}} \cdot \mathbf{v}_i). \end{aligned} \quad (\text{A14})$$

Since the numerator in the integrand (A9) is odd, under complex conjugation we have

$$F_I(-v) = F_I(+v)^*. \quad (\text{A15})$$

Hence, using Eq's. (A8) and (A15) can write

$$\begin{aligned} \frac{\pi v}{|k^2 \epsilon(k, \mathbf{v} \cdot \mathbf{k})|^2} \sum_i \beta_i e_i^2 \int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} f_i(\mathbf{p}_i) \delta(\hat{\mathbf{k}} \cdot \mathbf{v}_i - v) \\ = -\frac{1}{2i} \left\{ \frac{1}{k^2 + \kappa_e^2 + F_I(v)} - \frac{1}{k^2 + \kappa_e^2 + F_I(-v)} \right\}. \end{aligned} \quad (\text{A16})$$

## APPENDIX B: SCATTERING CORRECTIONS

The electron-ion energy exchange rate computed in Section 12 of BPS [1] was performed under quite general conditions, with no restriction on the masses, number densities, or temperatures of the plasma components, except that the plasma be fully ionized, non-degenerate, and weakly to moderately coupled (all mild restrictions in a hot, low- $Z$  plasma). Because of its generality, this result, which we shall present momentarily, is rather complicated. However, for most practical calculations, we can work in the high-temperature extreme-quantum limit and take advantage of the small electron-to-ion mass ratio. Under these conditions, we can use the Born approx-

imation for the two-body scattering amplitude, and the rate coefficient collapses to the simple expression,

$$\mathcal{C}_{ei} = \frac{\omega_I^2}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} (\beta_e e^2 n_e) \frac{1}{2} \left[ \ln \left\{ \frac{8T_e^2}{\hbar^2 \omega_e^2} \right\} - \gamma - 1 \right], \quad (\text{B1})$$

as previously quoted in (1.5). The purpose of this Appendix is to find the subleading quantum correction to Eq. (B1).

As noted in BPS, this subleading correction is of order

$$\eta_{ei}^2 \sim 27 \text{ eV}/T_e, \quad (\text{B2})$$

which, for most applications that we have in mind, is quite small. However, the leading electron degeneracy

effects are of order  $z_e \sim n_e a_0^3 (27 \text{ eV}/T_e)^{3/2}$ , which can be comparable in size to the subleading quantum correction. Both corrections are small compared to the leading-order contribution (B1). Therefore, in this Appendix, we can work in the non-degenerate limit, since degeneracy effects on top of the subleading quantum effects are smaller still. Consequently, our starting point will be the non-degenerate, but otherwise rather general, expression for the rate derived in Section 12 of BPS. We shall simplify this rate in favor of the more realistic case of light electrons and heavy ions, exhibiting this result in Eq. (B24), an expression that is valid to all orders in the quantum parameter  $\eta_{ei}$ . Although this expression is clear and compact, for the purposes of this paper, however, we only need the subleading  $\eta_{ei}^2$  term. This subleading correction is displayed in Eq. (B33).

The strength of the quantum effects associated with the scattering of two plasma species  $a$  and  $b$  is characterized by the dimensionless parameter

$$\bar{\eta}_{ab} = \frac{e_a e_b}{4\pi \hbar V_{ab}}, \quad (\text{B3})$$

where the square of the thermal velocity in this expression is defined by

$$V_{ab}^2 = \frac{1}{\beta_a m_a} + \frac{1}{\beta_b m_b}. \quad (\text{B4})$$

The extreme quantum limit, where formally  $\hbar \rightarrow \infty$ , is given by  $\bar{\eta}_{ab} \rightarrow 0$ ; while the extreme classical limit, where formally  $\hbar \rightarrow 0$ , is given by  $\bar{\eta}_{ab} \rightarrow \infty$ . The former case is equivalent to the Born approximation. In Section 12 of BPS, the energy exchange rate from an arbitrary plasma species  $a$  to another species  $b$ ,

$$\frac{d\mathcal{E}_{ab}}{dt} = -\mathcal{C}_{ab} (T_a - T_b), \quad (\text{B5})$$

was computed to all orders in the two-body quantum-scattering parameter  $\bar{\eta}_{ab}$ . It was found that the rate coefficient can be written as a sum of three terms, which, in the notation of BPS, reads

$$\mathcal{C}_{ab} = \mathcal{C}_{ab,R}^< + \left( \mathcal{C}_{ab,S}^C + \mathcal{C}_{ab}^{\Delta Q} \right), \quad (\text{B6})$$

where the last two terms have been grouped together for later convenience. These three terms are given by Eq's. (12.31), (12.25), and (12.50) respectively in BPS:

$$\mathcal{C}_{ab,R}^< = \frac{\kappa_a^2 \kappa_b^2}{2\pi} \left( \frac{\beta_a m_a}{2\pi} \right)^{1/2} \left( \frac{\beta_b m_b}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} dv v^2 e^{-\frac{1}{2}(\beta_a m_a + \beta_b m_b)v^2} \frac{i}{2\pi} \frac{F(v)}{\rho_{\text{total}}(v)} \ln \left\{ \frac{F(v)}{K^2} \right\}, \quad (\text{B7})$$

$$\mathcal{C}_{ab,S}^C = -\kappa_a^2 \kappa_b^2 \frac{(\beta_a m_a \beta_b m_b)^{1/2}}{(\beta_a m_a + \beta_b m_b)^{3/2}} \left( \frac{1}{2\pi} \right)^{3/2} \left[ \ln \left\{ \frac{e_a e_b}{4\pi} \frac{K}{4 m_{ab} V_{ab}^2} \right\} + 2\gamma \right], \quad (\text{B8})$$

and

$$\mathcal{C}_{ab}^{\Delta Q} = -\frac{1}{2} \kappa_a^2 \kappa_b^2 \frac{(\beta_a m_a \beta_b m_b)^{1/2}}{(\beta_a m_a + \beta_b m_b)^{3/2}} \left( \frac{1}{2\pi} \right)^{3/2} \int_0^{\infty} d\zeta e^{-\zeta/2} \left[ \text{Re} \psi \left( 1 + i \frac{\bar{\eta}_{ab}}{\zeta^{1/2}} \right) - \ln \left\{ \frac{\bar{\eta}_{ab}}{\zeta^{1/2}} \right\} \right]. \quad (\text{B9})$$

Since we are using Maxwell-Boltzmann statistics throughout this section, the Debye wave number of any plasma species, including electrons, is here determined by  $\kappa_b^2 = \beta_b e_b^2 n_b$ . The complex-valued function  $F(v)$  is defined by

$$F(v) = \int_{-\infty}^{\infty} du \frac{\rho_{\text{total}}(u)}{v - u + i\eta}, \quad (\text{B10})$$

where  $\rho_{\text{total}}(v)$  is the spectral weight,

$$\rho_{\text{total}}(v) = \sum_b \rho_b(v), \quad (\text{B11})$$

with

$$\rho_b(v) = \kappa_b^2 \sqrt{\frac{\beta_b m_b}{2\pi}} v \exp \left\{ -\frac{1}{2} \beta_b m_b v^2 \right\}. \quad (\text{B12})$$

This is similar to the function  $F_l$  introduced in the previous Appendix, except here the sum extends over all plasma species, including electrons. Note the dependence in Eq's. (B7) and (B8) on an unspecified parameter  $K$ , with the only restriction being that  $K$  has units of a wave-number. This is an artifact of the calculational procedure, and it was shown in BPS that the total rate  $\mathcal{C}_{ab}$  is indeed independent of  $K$ , as this parameter cancels in the sum between Eq's. (B7) and (B8). As



a matter of technical convenience, we will henceforth set  $K = \kappa_e$  throughout the rest of this Appendix [the simplified result (B18) only holds under this condition]. Finally, we should note that the reduced mass is determined by  $1/m_{ab} = 1/m_a + 1/m_b$ , and

$$\psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz} \quad (\text{B13})$$

is the logarithmic derivative of the gamma function.

Specializing to electrons and ions (in which  $a=e$  and  $b=i$ ), we can employ Eq. (B12) to write Eq. (B7) in the form

$$\mathcal{C}_{eI,R}^< \equiv \sum_i \mathcal{C}_{ei,R}^< = \frac{\kappa_e^2}{2\pi} \left( \frac{\beta_e m_e}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} dv v e^{-\frac{1}{2}\beta_e m_e v^2} \frac{i}{2\pi} \frac{\sum_i \rho_i(v)}{\rho_{\text{total}}(v)} F(v) \ln \left\{ \frac{F(v)}{\kappa_e^2} \right\} . \quad (\text{B14})$$

This expression greatly simplifies since the electron is so much lighter than the ions. In virtually all practical applications, the electron and ion temperatures are never excessively disparate, and we can therefore impose the mild restriction

$$\beta_e m_e \ll \beta_i m_i . \quad (\text{B15})$$

We will refer to the condition (B15) as the  $m_e \rightarrow 0$  limit, and the ratio  $\beta_e m_e / \beta_i m_i$  can then be used as a small dimensionless expansion parameter. For example, to leading order in this parameter we find

$$\frac{\sum_i \rho_i(v)}{\rho_{\text{total}}(v)} = 1 + \mathcal{O} \left( \frac{\beta_e m_e}{\beta_i m_i} \right)^{1/2} , \quad (\text{B16})$$

which allows us to express Eq. (B14) as

$$\mathcal{C}_{eI,R}^< = \frac{\kappa_e^2}{2\pi} \left( \frac{\beta_e m_e}{2\pi} \right)^{1/2} \frac{i}{2\pi} \int_{-\infty}^{\infty} dv v F(v) \ln \left\{ \frac{F(v)}{\kappa_e^2} \right\} \left[ 1 + \mathcal{O} \left( \frac{\beta_e m_e}{\beta_i m_i} \right)^{1/2} \right] . \quad (\text{B17})$$

We have omitted the exponential in the integrand of (B14), since the function  $F(v)$  provides enough convergence at large values of  $v$  to allow the  $m_e \rightarrow 0$  limit to be brought inside the integral. The analytic properties of  $F(v)$  allow us to perform the  $v$  integral in Eq. (B17) using contour integral techniques, in much the same manner as we did in the discussion following Eq. (7.9) in the text. The result is Eq. (12.44) of BPS, which reads

$$\beta_e m_e \ll \beta_i m_i : \quad \mathcal{C}_{eI,R}^< = -\frac{1}{2} \frac{\kappa_e^2}{2\pi} \left( \frac{\beta_e m_e}{2\pi} \right)^{1/2} \sum_i \omega_i^2 , \quad (\text{B18})$$

a much simpler expression indeed. For the electrons and ions we are considering, we can also drop the term of order  $\beta_e m_e / \beta_i m_i$  in Eq. (B3), allowing us to express the quantum parameter of Eq. (B4) as

$$\bar{\eta}_{ei} = \frac{Z_i e^2}{4\pi\hbar} \sqrt{\frac{m_e}{T_e}} = Z_i \bar{\eta}_e . \quad (\text{B19})$$

On occasion, we will express the quantum parameter in terms of the binding energy of the hydrogen atom

$$\epsilon_H = \frac{1}{2} \left( \frac{e^2}{4\pi} \right)^2 \frac{m_e}{\hbar^2} \simeq 13.606 \text{ eV} , \quad (\text{B20})$$

so that

$$\bar{\eta}_{ei}^2 = Z_i^2 \frac{2\epsilon_H}{T_e} . \quad (\text{B21})$$

We see that  $\bar{\eta}_{ei} \ll 1$  when  $T_e$  reaches the keV scale, illustrating that quantum corrections are important at high temperatures. Finally, we can drop terms of order  $\beta_e m_e / \beta_i m_i$  in the leading coefficients of Eq's. (B8) and (B9),

thereby allowing us to write

$$\begin{aligned} \beta_e m_e \ll \beta_i m_i : \\ \mathcal{C}_{eI,S}^C + \mathcal{C}_{eI}^{\Delta Q} = -\frac{\kappa_e^2}{2\pi} \left( \frac{\beta_e m_e}{2\pi} \right)^{1/2} \sum_i \omega_i^2 \left[ \ln \left\{ \frac{Z_i e^2}{4\pi} \frac{\kappa_e}{4T_e} \right\} + 2\gamma \right. \\ \left. + \frac{1}{2} \int_0^\infty d\zeta e^{-\zeta/2} \left( \operatorname{Re} \psi \left( 1 + i \frac{Z_i \bar{\eta}_e}{\zeta^{1/2}} \right) - \ln \left\{ \frac{Z_i \bar{\eta}_e}{\zeta^{1/2}} \right\} \right) \right] , \end{aligned} \quad (\text{B22})$$

Unlike Eq. (B18), which only holds for the sum over ion components, the result (B22) actually holds component by component. Performing the  $\zeta$  integral for the last term in Eq. (B22) gives

$$\begin{aligned} \beta_e m_e \ll \beta_i m_i : \\ \mathcal{C}_{eI,S}^C + \mathcal{C}_{eI}^{\Delta Q} = \frac{\kappa_e^2}{2\pi} \left( \frac{\beta_e m_e}{2\pi} \right)^{1/2} \sum_i \frac{\omega_i^2}{2} \left[ \ln \left\{ \frac{8m_e T_e}{\hbar^2 \kappa_e^2} \right\} - 3\gamma - \int_0^\infty d\zeta e^{-\zeta/2} \operatorname{Re} \psi \left( 1 + i \frac{\bar{\eta}_{ei}}{\zeta^{1/2}} \right) \right] . \end{aligned} \quad (\text{B23})$$

The rate  $\mathcal{C}_{eI}$  is given by adding Eq's. (B18) and (B23), which can be written as

$$\begin{aligned} \beta_e m_e \ll \beta_i m_i : \\ \mathcal{C}_{eI} = \frac{\kappa_e^2}{2\pi} \left( \frac{\beta_e m_e}{2\pi} \right)^{1/2} \frac{1}{2} \sum_i \omega_i^2 \left[ \ln \left\{ \frac{8m_e T_e}{\hbar^2 \kappa_e^2} \right\} - \gamma - 1 - \Delta_i(\bar{\eta}_{ei}) \right] , \end{aligned} \quad (\text{B24})$$

with

$$\Delta_i(\bar{\eta}_{ei}) = \int_0^\infty d\zeta e^{-\zeta/2} \left[ \operatorname{Re} \psi \left( 1 + i \frac{\bar{\eta}_{ei}}{\zeta^{1/2}} \right) + \gamma \right] . \quad (\text{B25})$$

This expression is accurate to leading and next-to-leading order in the plasma coupling, and to all orders in  $\bar{\eta}_{ei}$ , with no restriction on the temperature (apart from requiring the mild constraint (B15) and that the plasma coupling be small).

With the aid of Eq. (10.17) of BPS,

$$\operatorname{Re} \psi \left( 1 + i \bar{\eta} \zeta^{-1/2} \right) + \gamma = \sum_{k=1}^\infty \frac{1}{k} \frac{\bar{\eta}^2}{k^2 \zeta + \bar{\eta}^2} , \quad (\text{B26})$$

and writing

$$\frac{1}{\zeta + \bar{\eta}^2/k^2} = \frac{d}{d\zeta} \ln \left\{ \zeta + \frac{\bar{\eta}^2}{k^2} \right\} , \quad (\text{B27})$$

a partial integration now gives

$$\Delta_i(\bar{\eta}_{ei}) = \bar{\eta}_{ei}^2 \sum_{k=1}^\infty \frac{1}{k^3} \left[ \ln \left\{ \frac{k^2}{\bar{\eta}_{ei}^2} \right\} + \frac{1}{2} \int_0^\infty d\zeta e^{-\zeta/2} \ln \left\{ \zeta + \frac{\bar{\eta}_{ei}^2}{k^2} \right\} \right] . \quad (\text{B28})$$

In the latter form, we can easily extract the leading order term  $\bar{\eta}_{ei}$ , since we can use the limit

$$\begin{aligned} \bar{\eta}_{ei} \rightarrow 0 : \\ \frac{1}{2} \int_0^\infty d\zeta e^{-\zeta/2} \ln \left\{ \zeta + \frac{\bar{\eta}_{ei}^2}{k^2} \right\} \rightarrow \int_0^\infty d(\zeta/2) e^{-\zeta/2} \left( \ln \{ \zeta/2 \} + \ln 2 \right) \\ = -\gamma + \ln 2 . \end{aligned} \quad (\text{B29})$$

Using now

$$\zeta(3) = \sum_{k=1}^\infty \frac{1}{k^3} = 1.20205 \dots , \quad (\text{B30})$$

and

$$\zeta'(3) = - \sum_{k=1}^{\infty} \frac{1}{k^3} \ln k = -0.19812 \dots, \quad (\text{B31})$$

we can express the subleading quantum correction as

$$\Delta_i(\bar{\eta}_{ei}) \simeq \bar{\eta}_{ei}^2 \left\{ \zeta(3) \left[ \ln \left\{ \frac{2}{\bar{\eta}_{ei}^2} \right\} - \gamma \right] - 2 \zeta'(3) \right\}. \quad (\text{B32})$$

Using Eq. (B21), we can express the rate (B24) to leading and next-to-leading order in  $\bar{\eta}_{ei}$  as

$$\begin{aligned} \mathcal{C}_{eI} = & \frac{\omega_I^2}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} \kappa_e^2 \frac{1}{2} \left[ \ln \left\{ \frac{8 T_e^2}{\hbar^2 \omega_e^2} \right\} - \gamma - 1 \right] \\ & - \frac{1}{2\pi} \sqrt{\frac{\beta_e m_e}{2\pi}} \kappa_e^2 \frac{\epsilon_H}{T_e} \sum_i Z_i^2 \omega_i^2 \left[ \zeta(3) \left( \ln \left\{ \frac{T_e}{Z_i^2 \epsilon_H} \right\} - \gamma \right) - 2 \zeta'(3) \right]. \end{aligned} \quad (\text{B33})$$

The correction is of order  $\epsilon_H/T_e$ , which can be of comparable size to the leading degeneracy correction.

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